COMPATIBILITY BETWEEN SATAKE AND BERNSTEIN-TYPE ISOMORPHISMS IN CHARACTERISTIC p

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ABSTRACT. We study the center of the pro-p Iwahori-Hecke ring $\tilde{\mathrm{H}}_{\mathbb{Z}}$ of a connected split p-adic reductive group G. For k an algebraically closed field with characteristic p, we prove that the center of the k-algebra $\tilde{\mathrm{H}}_{\mathbb{Z}}\otimes_{\mathbb{Z}}k$ contains an affine semigroup algebra which is naturally isomorphic to the Hecke k-algebra $\mathcal{H}(\mathrm{G},\rho)$ attached to an irreducible smooth k-representation ρ of a given hyperspecial maximal compact subgroup of G. This isomorphism is obtained using the inverse Satake isomorphism defined in [24]. Finally, we apply this to study the "supersingular block" of the category of finite length $\tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$-modules and relate it to supersingular representations of G.

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1. Introduction

The Iwahori-Hecke ring of a split p-adic reductive group G is the convolution ring of $\mathbb{Z}$-valued functions with compact support in $\mathrm{I}\backslash\mathrm{G}/\mathrm{I}$ where I denotes an Iwahori subgroup of G. It is isomorphic to the quotient of the extended braid group associated to G by quadratic relations in the standard generators. If one replaces I by its pro-p Sylow subgroup $\tilde{\mathrm{I}}$, then one obtains the pro-p Iwahori-Hecke ring $\tilde{\mathrm{H}}_{\mathbb{Z}}$. In this article we study the center of $\tilde{\mathrm{H}}_{\mathbb{Z}}$. We are motivated by the smooth representation theory of G over an algebraically closed field k with characteristic p and subsequently will be interested in the k-algebra $\tilde{\mathrm{H}}_k := \tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$. We construct an isomorphism of k-algebras between a subring of the center of $\tilde{\mathrm{H}}_k$ and (generalizations of) spherical Hecke k-algebras by means of the (inverse) mod p Satake isomorphism defined in [24]. This result is the *compatibility between Bernstein and Satake isomorphisms* referred to in the title of this article. We then explore some consequences of this compatibility. In particular, we analyze the notion of supersingularity for Hecke modules.

1.1. **Framework and results.** Let $\mathfrak{F}$ be a nonarchimedean locally compact field with residue characteristic p and k an algebraic closure of the residue field. Let $\mathrm{G} := \mathbf{G}(\mathfrak{F})$ be the group of $\mathfrak{F}$-rational points of a connected reductive group $\mathbf{G}$ over $\mathfrak{F}$ which we assume to be $\mathfrak{F}$-split. In the semisimple building $\mathscr{X}$ of G, we choose and fix a chamber C which amounts to choosing an Iwahori subgroup I in G, and we denote by $\tilde{\mathrm{I}}$ the pro-p Sylow subgroup of I. This choice is unique up to conjugacy by an element of G. We consider the associated pro-p Iwahori Hecke ring $\tilde{\mathrm{H}}_{\mathbb{Z}}$ of $\mathbb{Z}$-valued functions with compact support in $\tilde{\mathrm{I}}\backslash\mathrm{G}/\tilde{\mathrm{I}}$ under convolution.

Since G is split, C has at least one hyperspecial vertex x_0 and we denote by K the associated maximal compact subgroup of G. Fix a maximal $\mathfrak{F}$-split torus T in G such that the corresponding apartment $\mathscr{A}$ in $\mathscr{X}$ contains C. The set $\mathrm{X}_*(\mathrm{T})$ of cocharacters of T is naturally equipped with an action of the finite Weyl group $\mathfrak{W}$. The choice of x_0 and of C imposes the choice of a positive Weyl chamber of $\mathscr{A}$ that is to say of a semigroup $\mathrm{X}_*^+(\mathrm{T})$ of dominant cocharacters of T.

1.1.1. *The complex case.* The structure of the spherical algebra $\mathbb{C}[\mathrm{K}\backslash\mathrm{G}/\mathrm{K}]$ of complex functions compactly supported on $\mathrm{K}\backslash\mathrm{G}/\mathrm{K}$ is understood thanks to the classical Satake isomosphism ([25], see also [12], [13])

$$s : \mathbb{C}[\mathrm{K}\backslash\mathrm{G}/\mathrm{K}] \xrightarrow{\simeq} (\mathbb{C}[\mathrm{X}_*(\mathrm{T})])^{\mathfrak{W}}.$$

On the other hand, the complex Iwahori-Hecke algebra $\mathrm{H}_{\mathbb{C}} := \mathbb{C}[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ contains a large commutative subalgebra $\mathcal{A}_{\mathbb{C}}$ defined as the image of the *Bernstein map* $\theta : \mathbb{C}[\mathrm{X}_*(\mathrm{T})] \hookrightarrow \mathrm{H}_{\mathbb{C}}$ which depends on the choice of the dominant Weyl chamber (see [21, 3.2]). The algebra $\mathrm{H}_{\mathbb{C}}$ is free of finite rank over $\mathcal{A}_{\mathbb{C}}$ and its center $\mathcal{Z}(\mathrm{H}_{\mathbb{C}})$ is contained in $\mathcal{A}_{\mathbb{C}}$. Furthermore, the map θ yields an isomorphism

$$b : \mathbb{C}[\mathrm{X}_*(\mathrm{T})])^{\mathfrak{W}} \xrightarrow{\simeq} \mathcal{Z}(\mathrm{H}_{\mathbb{C}}).$$

This was proved by Bernstein ([21, 3.5], see also [13, Theorem 2.3]). By [10, Corollary 3.1] and [13, Proposition 10.1], the *Bernstein isomophism* b is compatible with s in the sense that the composition $(e_{\mathrm{K}} \star .)b$ is an inverse for s, where $(e_{\mathrm{K}} \star .)$ is the convolution by the characteristic function of K.

1.1.2. *Bernstein and Satake isomorphisms in characteristic p.* After defining an integral version of the complex Bernstein map, Vignéras gave in [31] a basis for the center of $\tilde{\mathrm{H}}_{\mathbb{Z}}$ and proved that $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is noetherian and finitely generated over its center. In the first section of this article, we define a subring $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathbb{Z}})$ of the center of $\tilde{\mathrm{H}}_{\mathbb{Z}}$ over which $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is still finitely generated. The vertex x_0 is chosen to be hyperspecial in the current article. In Proposition 2.6 we prove that $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathbb{Z}})$ is not affected by the choice of another apartment containing C and of another hyperspecial vertex of C as long as it is conjugate to x_0. In particular, if $\mathbf{G}$ is of adjoint type or $\mathbf{G} = \mathrm{GL}_n$, then $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathbb{Z}})$ depends only on the choice of the uniformizer ϖ.

The image of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathbb{Z}})$ in $\tilde{\mathrm{H}}_k = \tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$ is denoted by $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ and we prove that it has a structure of affine semigroup algebra. More precisely, we have an isomorphism of k-algebras (Proposition 2.8)

$$k[\mathrm{X}_*^+(\mathrm{T})] \xrightarrow{\sim} \mathcal{Z}^\circ(\tilde{\mathrm{H}}_k) \subseteq \tilde{\mathrm{H}}_k. \tag{1.1}$$

By the main theorem in [16] (and in [24]), this makes $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ isomorphic to the algebra $\mathcal{H}(\mathrm{G}, \rho)$ of any irreducible smooth k-representation ρ of K. Note that when ρ is the k-valued trivial representation $\mathbf{1}_{\mathrm{K}}$ of K, ones retrieves the convolution algebra $k[\mathrm{K}\backslash\mathrm{G}/\mathrm{K}] = \mathcal{H}(\mathrm{G}, \mathbf{1}_{\mathrm{K}})$.

In [24], we constructed an isomorphism

$$\mathcal{T} : k[\mathrm{X}_*^+(\mathrm{T})] \xrightarrow{\simeq} \mathcal{H}(\mathrm{G}, \rho). \tag{1.2}$$

In the current article, we prove the following theorem.

Theorem 1.1. *We have a commutative diagram of isomorphisms of k-algebras*

$$\begin{array}{ccc} k[\mathrm{X}_*^+(\mathrm{T})] & \xrightarrow{(1.1)} & \mathcal{Z}^\circ(\tilde{\mathrm{H}}_k) \\ \| & & \downarrow \\ k[\mathrm{X}_*^+(\mathrm{T})] & \xrightarrow{\mathcal{T}} & \mathcal{H}(\mathrm{G}, \rho) \end{array} \tag{1.3}$$

where the vertical arrow on the right hand side is the natural morphism of k-algebras described in Section 4.

The isomorphism $\mathcal{T}$ was constructed in [24] by means of generalized integral Bernstein maps, as are the subring $\mathscr{Z}^\circ(\tilde{\mathrm{H}}_k)$ and the map (1.1) in the current article. By analogy with the complex case, we can see the map (1.1) as an isomorphism à la Bernstein in characteristic p. Then the above commutative diagram can be interpreted as a statement of compatibility between Satake and Bernstein isomorphisms in characteristic p. Note that under the hypothesis that the derived subgroup of $\mathbf{G}$ is simply connected, it is proved in [24] that $\mathcal{T}$ is the inverse of the mod p Satake isomorphism defined by Herzig in [16].

If we had worked with the Iwahori-Hecke algebra $k[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$, the analogous of $\mathscr{Z}^\circ(\tilde{\mathrm{H}}_k)$ would actually be the whole center of $k[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ (which obviously does not depend on any choice). We prove:

Theorem (Theorem 2.12)**.** *The center of the Iwahori-Hecke k-algebra* $k[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ *is isomorphic to* $k[\mathrm{X}_*^+(\mathrm{T})]$.

1.1.3. *Generalized integral Bernstein maps.* One ingredient of the construction of $\mathcal{T}$ in [24] and of the proof of Theorem 1.1 is the definition of $\mathbb{Z}$-linear injective maps

$$\mathcal{B}_F^\sigma : \mathbb{Z}[\tilde{\mathrm{X}}_*(\mathrm{T})] \to \tilde{\mathrm{H}}_{\mathbb{Z}}$$

defined on the group ring of the (extended) cocharacters $\tilde{\mathrm{X}}_*(\mathrm{T})$, and which are multiplicative when restricted to the semigroup ring of any chosen Weyl chamber of $\tilde{\mathrm{X}}_*^+(\mathrm{T})$ (see 1.2.5 for the definition of $\tilde{\mathrm{X}}_*(\mathrm{T})$). The image of $\mathcal{B}_F^\sigma$ happens to be a commutative subring of $\tilde{\mathrm{H}}_{\mathbb{Z}}$ which we denote by $\mathcal{A}_F^\sigma$. The parameter σ is a sign and F is a standard facet, meaning a facet of C containing x_0 in its closure. The choice of F corresponds to the choice of a Weyl chamber in $\mathscr{A}$: for example if $F = C$ (resp. x_0) the corresponding Weyl chamber is the dominant (resp. antidominant) one.

The maps $\mathcal{B}_F^\sigma$ are called *integral Bernstein maps* because they are generalization of the Bernstein map θ mentioned in 1.1.1. In the complex case, it is customary to consider either θ (which is constructed using the dominant chamber), or θ^- which is constructed using the antidominant chamber (see the dicussion in the introduction of [14] for example). By a result by Bernstein ([20]) a basis for the center of $\mathrm{H}_{\mathbb{C}}$ is given by the central Bernstein functions

$$\sum_{\lambda' \in \mathcal{O}} \theta(\lambda')$$

where $\mathcal{O}$ ranges over the $\mathfrak{W}$-orbits in $\mathrm{X}_*(\mathrm{T})$. We refer to [13] for the geometric interpretation of these functions. It is natural to ask whether using θ^- instead of θ in the previous formula yields the same central element in $\mathrm{H}_{\mathbb{C}}$. The answer is yes (see [14, 2.2.2]). The proof is based on [20,

Corollary 8.8] and relies on the combinatorics of the Kazhdan-Lusztig polynomials. Note that there is no such theory (yet) for the complex pro-p Iwahori-Hecke algebra.

Integral (and pro-p) versions of θ and θ^- for the ring $\tilde{\mathrm{H}}_{\mathbb{Z}}$ were defined in [31]. In our language they correspond respectively to $\mathcal{B}^+_C = \mathcal{B}^-_{x_0}$ and $\mathcal{B}^+_{x_0} = \mathcal{B}^-_C$. It is also proved in [31] that a $\mathbb{Z}$-basis for the center of $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is given by

$$\sum_{\lambda' \in \mathcal{O}} \mathcal{B}^+_C(\lambda')$$

where $\mathcal{O}$ ranges over the $\mathfrak{W}$-orbits in $\tilde{\mathrm{X}}_*(\mathrm{T})$. It is now natural to ask whether the element above is the same if **a/** we use $+$ instead of $-$, and if more generally **b/** we use any standard facet F and any sign σ. We prove:

Lemma (Lemma 3.4)**.** *The element*

$$\sum_{\lambda' \in \mathcal{O}} \mathcal{B}^\sigma_F(\lambda')$$

in $\tilde{\mathrm{H}}_{\mathbb{Z}}$ *does not depend on the choice of the standard facet* F *and of the sign* σ*.*

This lemma is followed in the body of the article by a flower-like drawing which is meant to illustrate the fact that the center of $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is contained in the intersection of all the commutative rings $\mathcal{A}^\sigma_F$ for F a standard facet and σ a sign.

To prove the lemma, we first answer positively question **a/** above. We then study and exploit the behavior of the Bernstein integral maps upon a process of parabolic induction. In passing we also consider question **a/** in the k-algebra $\tilde{\mathrm{H}}_k$ in the case when G is semisimple, and we suggest a link between such questions and the duality for finite length $\tilde{\mathrm{H}}_k$-modules defined in [23] (see Proposition 3.3).

1.1.4. In Section 5, we define and study a natural topology on $\tilde{\mathrm{H}}_k$ which depends only on the conjugacy class of x_0. It is the $\mathfrak{J}$-adic topology where $\mathfrak{J}$ is a natural monomial ideal of the affine semigroup algebra $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$.

We define the supersingular block of the category of finite length (left) $\tilde{\mathrm{H}}_k$-modules to be the full subcategory of the modules that are continuous for the $\mathfrak{J}$-adic topology on $\tilde{\mathrm{H}}_k$ (Proposition-Definition 5.9). This definition extends the notion of supersingularity for irreducible $\tilde{\mathrm{H}}_k$-modules given in [31]. Thus defined, the supersingular block depends only on the conjugacy class of x_0, and therefore in the case where $\mathbf{G}$ is of adjoint type or $\mathbf{G} = \mathrm{GL}_n$ it is independent of all the choices.

We prove (Proposition 5.12) that if a finite length module is in the supersingular block then it contains a character for the affine subalgebra of the functions with support in the subgroup of G generated by all parahoric subgroups. This extends [22, Theorem 7.3] that dealt with the case of GL_n. In the case of GL_n, the converse statement is true by [31, Theorem 5].

We conjecture that the statement of Proposition 5.12 is an equivalence for a general $\mathfrak{F}$-split group and hope to investigate this question in future work.

1.1.5. In 5.5 we consider an admissible irreducible smooth k-representation π of G. In the case where the derived subgroup of $\mathbf{G}$ is simply connected, we use the fact that (1.2) is the inverse of the mod p Satake isomorphism defined in [16] to prove that if π is supersingular, then

$$\pi \text{ is a quotient of } \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1 / \mathfrak{I}\, \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1. \tag{1.4}$$

Note that the above condition depends only on the conjugacy class of x_0. In the case when $\mathrm{G} = \mathrm{GL}_n(\mathfrak{F})$ and $\mathfrak{F}$ is a finite extension of $\mathbb{Q}_p$, we use the classification of the nonsupersingular representations obtained in [17], the work on special representations in [11], and our Lemma 3.4, to prove that the condition (1.4) is in fact a characterization of the supersingular representations. We expect that there is a direct proof of this result (and a version of it in the case of a general $\mathfrak{F}$-split group), that does not use the classification of the nonsupersingular representations.

1.2. **Notation and preliminaries.** We choose the valuation $val_{\mathfrak{F}}$ on $\mathfrak{F}$ normalized by $val_{\mathfrak{F}}(\varpi) = 1$ where ϖ is the chosen uniformizer. The ring of integers of $\mathfrak{F}$ is denoted by $\mathfrak{O}$ and its residue field by $\mathbb{F}_q$ where q is a power of the prime number p. Recall that k denotes an algebraic closure of $\mathbb{F}_q$. Let $\mathbf{G}_{x_0}$ and $\mathbf{G}_C$ denote the Bruhat-Tits group schemes over $\mathfrak{O}$ whose $\mathfrak{O}$-valued points are K and I respectively. Their reductions over the residue field $\mathbb{F}_q$ are denoted by $\overline{\mathbf{G}}_{x_0}$ and $\overline{\mathbf{G}}_C$. Note that $\mathrm{G} = \mathbf{G}_{x_0}(\mathfrak{F}) = \mathbf{G}_C(\mathfrak{F})$. By [29, 3.4.2, 3.7 and 3.8], $\overline{\mathbf{G}}_{x_0}$ is connected reductive and $\mathbb{F}_q$-split. Therefore we have $\mathbf{G}_C^{\circ}(\mathfrak{O}) = \mathbf{G}_C(\mathfrak{O}) = \mathrm{I}$ and $\mathbf{G}_{x_0}^{\circ}(\mathfrak{O}) = \mathbf{G}_{x_0}(\mathfrak{O}) = \mathrm{K}$. Denote by K_1 the pro-unipotent radical of K. The quotient K/K_1 is isomorphic to $\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$. The Iwahori subgroup I is the preimage in K of the $\mathbb{F}_q$-rational points of a Borel subgroup $\overline{\mathbf{B}}$ with Levi decomposition $\overline{\mathbf{B}} = \overline{\mathbf{T}}\,\overline{\mathbf{N}}$. The pro-$p$ Iwahori subgroup $\tilde{\mathrm{I}}$ is the preimage in I of $\overline{\mathbf{N}}(\mathbb{F}_q)$. The preimage of $\overline{\mathbf{T}}(\mathbb{F}_q)$ is the the maximal compact subgroup T^0 of T. Note that $\mathrm{T}^0/\mathrm{T}^1 = \mathrm{I}/\tilde{\mathrm{I}} = \overline{\mathbf{T}}(\mathbb{F}_q)$ where $\mathrm{T}^1 := \mathrm{T}^0 \cap \tilde{\mathrm{I}}$.

1.2.1. *Affine root datum.* To the choice of T is attached the root datum $(\Phi, \mathrm{X}^*(\mathrm{T}), \check{\Phi}, \mathrm{X}_*(\mathrm{T}))$. This root system is reduced because the group $\mathbf{G}$ is $\mathfrak{F}$-split. We denote by $\mathfrak{W}$ the finite Weyl group $N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$, quotient by T of the normalizer of T. Recall that $\mathscr{A}$ denotes the apartment of the semisimple building attached to T ([29] and [27, I.1], and we follow the notations of [24, 2.2]). We denote by $\langle\, .\, ,\, .\, \rangle$ the perfect pairing $\mathrm{X}_*(\mathrm{T}) \times \mathrm{X}^*(\mathrm{T}) \to \mathbb{Z}$. We will call coweights the elements in $\mathrm{X}_*(\mathrm{T})$. We identify $\mathrm{X}_*(\mathrm{T})$ with the subgroup T/T^0 of the extended Weyl group $\mathrm{W} = N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}^0$ as in [29, I.1] and [27, I.1]: to an element $g \in \mathrm{T}$ corresponds a vector $\nu(g) \in \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(\mathrm{T})$ defined by

$$\langle \nu(g),\, \chi \rangle = -\, val_{\mathfrak{F}}(\chi(g)) \qquad \text{for any } \chi \in \mathrm{X}^*(\mathrm{T}). \tag{1.5}$$

and ν induces the required isomorphism $\mathrm{T}/\mathrm{T}^0 \cong \mathrm{X}_*(\mathrm{T})$. The group T/T^0 acts by translation on $\mathscr{A}$ via ν. The actions of $\mathfrak{W}$ and T/T^0 combine into an action of W on $\mathscr{A}$ as recalled in

[27, page 102]. Since x_0 is a special vertex of the building, W is isomorphic to the semidirect product $\mathfrak{W} \ltimes \mathrm{X}_*(\mathrm{T})$ where we see $\mathfrak{W}$ as the fixator in W of any point lifting x_0 in the extended apartment ([29, 1.9]). A coweight λ will sometimes be denoted by e^λ to underline that we see it as an element in W, meaning as a translation on $\mathscr{A}$.

Denote by Φ_{aff} the set of affine roots. The choice of the chamber C implies in particular the choice of the positive affine roots Φ^+_{aff} taking nonnegative values on C. The choice of x_0 as an origin of $\mathcal{A}$ implies that we identify the affine roots taking value zero at x_0 with Φ. We set $\Phi^+ := \Phi^+_{aff} \cap \Phi$ and $\Phi^- = -\Phi^+$. The affine roots can be described the following way: $\Phi_{aff} = \Phi \times \mathbb{Z} = \Phi^+_{aff} \coprod \Phi^-_{aff}$ where

$$\Phi^+_{aff} := \{(\alpha, r),\ \alpha \in \Phi,\ r > 0\} \cup \{(\alpha, 0),\ \alpha \in \Phi^+\}.$$

Let Π be the basis for Φ^+: it is the set of simple roots. The finite Weyl group $\mathfrak{W}$ is a Coxeter system with generating set $S := \{s_\alpha,\ \alpha \in \Pi\}$ where s_α denotes the (simple) reflection at the hyperplane $\langle\, .\,, \alpha\rangle = 0$. Denote by $\preceq$ the partial ordering on $\mathrm{X}^+_*(\mathrm{T})$ associated to Π. Let Π_m be the set of roots in Φ that are minimal elements for $\preceq$. Define the set of simple affine roots by $\Pi_{aff} := \{(\alpha, 0),\ \alpha \in \Pi\} \cup \{(\alpha, 1),\ \alpha \in \Pi_m\}$. Identifying α with $(\alpha, 0)$, we consider Π a subset of Π_{aff}. For $A \in \Pi_{aff}$, denote by s_A the following associated reflection: $s_A = s_\alpha$ if $A = (\alpha, 0)$ and $s_A = s_\alpha e^{\check{\alpha}}$ if $A = (\alpha, 1)$. The action of W on the coweights induces an action on the set of affine roots: W acts on Φ_{aff} by $we^\lambda : (\alpha, r) \mapsto (w\alpha,\ r - \langle \lambda, \alpha\rangle)$ where we denote by $(w, \alpha) \mapsto w\alpha$ the natural action of $\mathfrak{W}$ on Φ. The length on the Coxeter system $(\mathfrak{W}, S)$ extends to W in such a way that the length $\ell(w)$ of $w \in \mathrm{W}$ is the number of affine roots $A \in \Phi^+_{aff}$ such that $w(A) \in \Phi^-_{aff}$. It satisfies the following formula, for every $A \in \Pi_{aff}$ and $w \in \mathrm{W}$:

$$\ell(ws_A) = \begin{cases} \ell(w) + 1 & \text{if } w(A) \in \Phi^+_{aff}, \\ \ell(w) - 1 & \text{if } w(A) \in \Phi^-_{aff}. \end{cases} \tag{1.6}$$

The affine Weyl group is defined as the subgroup W_{aff} of W generated by $S_{aff} := \{s_A,\ A \in \Phi_{aff}\}$. The length function ℓ restricted to W_{aff} coincides with the length function of the Coxeter system $(\mathrm{W}_{aff}, S_{aff})$ ([4, V.3.2 Thm 1(i)]). Recall ([21, 1.5]) that W_{aff} is a normal subgroup of W: the set Ω of elements with length zero is an abelian subgroup of W and W is the semidirect product $\mathrm{W} = \Omega \ltimes \mathrm{W}_{aff}$. The length ℓ is constant on the double cosets of W mod Ω. In particular Ω normalizes S_{aff}.

The extended Weyl group W is equipped with a partial order $\leq$ that extends the Bruhat order on W_{aff}. By definition, given $w = \omega w_{aff}$, $w = \omega' w'_{aff} \in \Omega \ltimes \mathrm{W}_{aff}$, we have $w \leq w'$ if $\omega = \omega'$ and $w_{aff} \leq w'_{aff}$ in the Bruhat order on W_{aff} (see for example [13, 2.1]).

We fix a lift $\hat{w} \in N_{\mathrm{G}}(\mathrm{T})$ for any $w \in \mathrm{W}$. By Bruhat decomposition, G is the disjoint union of all $\mathrm{I}\hat{w}\mathrm{I}$ for $w \in \mathrm{W}$.

1.2.2. *Orientation character.* The stabilizer of the chamber C in W is Ω. We define as in [23, 3.1] the orientation character $\epsilon_C : \Omega \to \{\pm 1\}$ of C by setting $\epsilon_C(\omega) = +1$, resp. -1, if ω preserves, resp. reverses, a given orientation of C. Since $\mathrm{W}/\mathrm{W}_{aff} = \Omega$ we can see ϵ_C as a character of W trivial on W_{aff}. By definition of the Bruhat order on W, we have $\epsilon_C(w) = \epsilon_C(w')$ for $w, w' \in \mathrm{W}$ satisfying $w \leq w'$.

On the other hand, the extended Weyl group acts by affine isometries on the Euclidean space $\mathscr{A}$. We therefore have a determinant map $\det : \mathrm{W} \to \{\pm 1\}$ which is trivial on $\mathrm{X}_*(\mathrm{T})$. An orientation of C is a choice of a cyclic ordering of its set of vertices (in the geometric realization of $\mathscr{A}$). Therefore, $\det(\omega)$ is the signature of the permutation of the vertices of C induced by $\omega \in \Omega$ and $\det(\omega) = \epsilon_C(\omega)$.

Lemma 1.2. (1) *For $w \in \mathrm{W}_{aff}$ we have* $\det(w) = (-1)^{\ell(w)}$.

(2) *For $\lambda \in \mathrm{X}_*(\mathrm{T})$, we have $\epsilon_C(w) = (-1)^{\ell(e^\lambda)}$ for any $w \in \mathrm{W}$ such that $w \leq e^\lambda$.*

Proof. The first point comes from the fact that $\det(s) = -1$ for $s \in S_{aff}$. For the second one, by definition of the Bruhat order, it is enough to prove that $\epsilon_C(e^\lambda) = (-1)^{\ell(e^\lambda)}$ for $\lambda \in \mathrm{X}_*(\mathrm{T})$. Decompose $e^\lambda = \omega w_{aff}$ with $w \in \mathrm{W}_{aff}$ and $\omega \in \Omega$. Recall that ω has length zero. Since ϵ_C is trivial on W_{aff}, we have $\epsilon_C(e^\lambda) = \epsilon_C(\omega) = \det(\omega)$. Since $\det(e^\lambda) = 1$ we have $\det(\omega) = \det(w_{aff}) = (-1)^{\ell(w_{aff})} = (-1)^{\ell(e^\lambda)}$.

□

1.2.3. *Distinguished cosets representatives.*

Proposition 1.3. i. *The set $\mathcal{D}$ of all elements $d \in \mathrm{W}$ satisfying $d(\Phi^+) \subset \Phi^+_{aff}$ is a system of representatives of the left cosets $\mathrm{W}/\mathfrak{W}$. It satisfies*

$$\ell(dw) = \ell(d) + \ell(w) \text{ for any } w \in \mathfrak{W} \text{ and } d \in \mathcal{D}. \tag{1.7}$$

In particular, d is the unique element with minimal length in $d\mathfrak{W}$.

ii. *An element $d \in \mathcal{D}$ can be written uniquely $d = we^{-\lambda}$ with $\lambda \in \mathrm{X}^+_*(\mathrm{T})$ and $w \in \mathfrak{W}$. We then have $\ell(e^{-\lambda}) = \ell(w^{-1}) + \ell(d)$.*

iii. *For $s \in S_{aff}$ and $d \in \mathcal{D}$, we are in one of the following situations:*
- *$\ell(sd) = \ell(d) - 1$ in which case $sd \in \mathcal{D}$.*
- *$\ell(sd) = \ell(d) + 1$ in which case either $sd \in \mathcal{D}$ or $sd \in d\mathfrak{W}$.*

Proof. This proposition is proved in [22, Lemma 2.6, Prop. 2.7] in the case of $\mathrm{G} = \mathrm{GL}_n(\mathfrak{F})$. It is checked in [23, Prop. 4.6] that it remains valid for a general split reductive group (see also [24, Prop. 2.2] for ii), except for point iii when $s \in S_{aff} - S$. We check here that the argument goes through. Let $s \in S_{aff}$ and A the corresponding affine root. Let $d \in \mathcal{D}$ and suppose that $sd \notin \mathcal{D}$, then there is $\beta \in \Pi$ such that $sd\beta \in \Phi^-_{aff}$ while $d\beta \in \Phi^+_{aff}$. It implies that $d\beta = A$ which in particular ensures that $d^{-1}A \in \Phi^+_{aff}$ and therefore $\ell(sd) = \ell(d) + 1$. Furthermore, $d^{-1}sd = s_{d^{-1}A} = s_\beta \in \mathfrak{W}$.

□

Recall that there is an action of the group G on the semisimple building $\mathscr{X}$ recalled in [27, page 104] that extends the action of $N_{\mathrm{G}}(\mathrm{T})$ on the standard apartment. For F a standard facet, we denote by $\mathcal{P}_F^{\dagger}$ the stabilizer of F in G.

Proposition 1.4. i. *The Iwahori subgroup* I *acts transitively on the apartments of $\mathscr{X}$ containing C.*

ii. *The stabilizer $\mathcal{P}_{x_0}^{\dagger}$ of x_0 acts transitively on the chambers of $\mathscr{X}$ containing x_0 in their closure.*

iii. *A* G*-conjugate of x_0 in the closure of C is a $\mathcal{P}_C^{\dagger}$-conjugate of x_0.*

Proof. Point i is [6, 4.6.28]. For ii, we first consider C' a chamber of $\mathscr{A}$ containing x_0 in its closure. The group W acting transitively on the chambers of $\mathscr{A}$, there is $d \in \mathcal{D}$ and $w_0 \in \mathfrak{W}$ such that $C' = w_0 d^{-1} C$ and C contains dx_0 in its closure. By [23, Proposition 4.13 i.], it implies that $dC = C$ so that $C' = w_0 C$ or $C' = \hat{w}_0 C$ where $\hat{w}_0 \in \mathrm{K} \cap N_G(\mathrm{T})$ denotes a lift for w_0. Now let C'' be a chamber of $\mathscr{X}$ containing x_0 in its closure. By [5, Corollaire 2.2.6], there is $k \in \mathcal{P}_{x_0}^{\dagger}$ such that kC'' is in $\mathscr{A}$. Applying the previous observation, C'' is a $\mathcal{P}_{x_0}^{\dagger}$-conjugate of C. Lastly, let gx_0 (with $g \in \mathrm{G}$) be a conjugate of x_0 in the closure of C. By ii, the chamber $g^{-1}C$ is of the form kC for $k \in \mathcal{P}_{x_0}^{\dagger}$ which implies that $gk \in \mathcal{P}_C^{\dagger}$ and gx_0 is a $\mathcal{P}_C^{\dagger}$-conjugate of x_0.

□

Remark 1.5. By [23, Lemma 4.9], $\mathcal{P}_C^{\dagger}$ is the disjoint union of all $\mathrm{I}\hat{\omega}\mathrm{I} = \hat{\omega}\mathrm{I}$ for $\omega \in \Omega$. Therefore, a G-conjugate of x_0 in the closure of C is a $\mathcal{P}_C^{\dagger} \cap N_{\mathrm{G}}(\mathrm{T})$-conjugate of x_0.

1.2.4. *Weyl chambers.* The set of dominant coweights $\mathrm{X}_*^+(\mathrm{T})$ is the set of all $\lambda \in \mathrm{X}_*(\mathrm{T})$ such that $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+$. It is called the dominant chamber. Its opposite is the antidominant chamber. A coweight λ such that $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Phi^+$ is called strongly dominant. By [7, Lemma 6.14], strongly dominant elements do exist.

We call a facet F of $\mathscr{A}$ standard if it is a facet of C containing x_0 in its closure. Attached to a standard facet F is the set Φ_F of all roots taking value zero on F and the subgroup $\mathfrak{W}_F$ of $\mathfrak{W}$ generated by the simple reflections stabilizing F. Let $\Phi_F^+ := \Phi^+ \cap \Phi_F$ and $\Phi_F^- := \Phi^- \cap \Phi_F$. Define the following Weyl chamber in $\mathrm{X}_*(\mathrm{T})$ as in [24, 4.1.1]:

$$\mathscr{C}^+(F) = \{\lambda \in \mathrm{X}_*(\mathrm{T}) \text{ such that } \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in (\Phi^+ - \Phi_F^+) \cup \Phi_F^- \}$$

and its opposite $\mathscr{C}^-(F) = -\mathscr{C}^+(F)$. They are respectively the images of the dominant and antidominant chambers by the longest element in $\mathfrak{W}_F$.

By Gordan's Lemma ([19, p. 7]), a Weyl chamber is finitely generated as a semigroup.

1.2.5. We follow the notations of [24, 2.2.2, 2.2.3]. Recall that T^1 is the pro-p Sylow subgroup of T^0. We denote by $\tilde{\mathrm{W}}$ the quotient of $N_{\mathrm{G}}(\mathrm{T})$ by T^1 and obtain the exact sequence

$$0 \to \mathrm{T}^0/\mathrm{T}^1 \to \tilde{\mathrm{W}} \to \mathrm{W} \to 0.$$

The group $\tilde{\mathrm{W}}$ parametrizes the double cosets of G modulo $\tilde{\mathrm{I}}$. We fix a lift $\hat{w} \in N_{\mathrm{G}}(\mathrm{T})$ for any $w \in \tilde{\mathrm{W}}$ and denote by τ_w the characteristic function of the double coset $\tilde{\mathrm{I}}\hat{w}\tilde{\mathrm{I}}$. The set of all $(\tau_w)_{w\in\tilde{\mathrm{W}}}$ is a $\mathbb{Z}$-basis for $\tilde{\mathrm{H}}_{\mathbb{Z}}$ which was defined in the introduction to be the convolution ring of $\mathbb{Z}$-valued functions with compact support in $\tilde{\mathrm{I}}\backslash\mathrm{G}/\tilde{\mathrm{I}}$. For $g \in \mathrm{G}$, we will also use the notation τ_g for the characteristic function of the double coset $\tilde{\mathrm{I}}g\tilde{\mathrm{I}}$.

For Y a subset of W, we denote by $\tilde{Y}$ its preimage in $\tilde{\mathrm{W}}$. In particular, we have the preimage $\tilde{\mathrm{X}}_*(\mathrm{T})$ of $\mathrm{X}_*(\mathrm{T})$. As well as those of $\mathrm{X}_*(\mathrm{T})$, its elements will be denoted by λ or e^λ and called coweights. For $\alpha \in \Phi$, we inflate the function $\langle . \, , \alpha\rangle$ defined on $\mathrm{X}_*(\mathrm{T})$ to $\tilde{\mathrm{X}}_*(\mathrm{T})$. We still call *dominant coweights* the elements in the preimage $\tilde{\mathrm{X}}_*^+(\mathrm{T})$ of $\mathrm{X}_*^+(\mathrm{T})$. For σ a sign and F a standard facet, we consider the preimage of $\mathscr{C}^\sigma(F)$ in $\tilde{\mathrm{X}}_*(\mathrm{T})$ and we still denote it by $\mathscr{C}^\sigma(F)$.

The length function ℓ on W pulls back to a length function ℓ on $\tilde{\mathrm{W}}$ ([31, Prop. 1]). For $u, v \in \tilde{\mathrm{W}}$ we write $u \leq v$ (resp. $u < v$) if their projections $\bar{u}$ and $\bar{v}$ in W satisfy $\bar{u} \leq \bar{v}$ (resp. $\bar{u} < \bar{v}$).

1.2.6. We emphasize the following remark which will be important for the definition of the subalgebra $\mathscr{Z}^\circ(\tilde{\mathrm{H}}_{\mathbb{Z}})$ of the center of $\tilde{\mathrm{H}}_{\mathbb{Z}}$.

For $\lambda \in \mathrm{X}_*^+(\mathrm{T})$, the element $\lambda(\varpi^{-1}) \in N_{\mathrm{G}}(\mathrm{T})$ is a lift for e^λ seen in W by our convention (1.5). The map

$$\lambda \in \mathrm{X}_*(\mathrm{T}) \to [\lambda(\varpi^{-1}) \bmod \mathrm{T}^1] \in \tilde{\mathrm{X}}_*(\mathrm{T}) \tag{1.8}$$

is a $\mathfrak{W}$-equivariant splitting for the exact sequence of abelian groups

$$0 \longrightarrow \mathrm{T}^0/\mathrm{T}^1 \longrightarrow \tilde{\mathrm{X}}_*(\mathrm{T}) \longrightarrow \mathrm{X}_*(\mathrm{T}) \longrightarrow 0. \tag{1.9}$$

We will identify $\mathrm{X}_*(\mathrm{T})$ with its image in $\tilde{\mathrm{X}}_*(\mathrm{T})$ *via* (1.8). Note that this identification depends on the choice of the uniformizer ϖ.

Remark 1.6. We have the decomposition of $\tilde{\mathrm{W}}$ as a semidirect product $\tilde{\mathrm{W}} = \tilde{\mathfrak{W}} \ltimes \mathrm{X}_*(\mathrm{T})$ where $\tilde{\mathfrak{W}}$ denotes the preimage of $\mathfrak{W}$ in $\tilde{\mathrm{W}}$.

1.2.7. *Pro-p Hecke rings.* The product in $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is described in [31, Theorem 1]. It is given by *quadratic relations* and *braid relations*. Stating the quadratic relations in the generic pro-p Hecke ring requires some more notations. Since we are only going to use them in $\tilde{\mathrm{H}}_k$ where they have a simpler form, we postpone their description to 1.2.8. We recall here the braid relations:

$$\tau_{ww'} = \tau_w\tau_{w'} \text{ for } w, w' \in \tilde{\mathrm{W}} \text{ satisfying } \ell(ww') = \ell(w) + \ell(w'). \tag{1.10}$$

The functions in $\tilde{\mathrm{H}}_{\mathbb{Z}}$ with support in the subgroup of G generated by all parahoric subgroups form a subring $\tilde{\mathrm{H}}_{\mathbb{Z}}^{aff}$ called the affine subring. It has $\mathbb{Z}$-basis the set of all τ_w for w in the preimage $\tilde{\mathrm{W}}_{aff}$ of W_{aff} in $\tilde{\mathrm{W}}$ (see for example [23, 4.5]). It is generated by all τ_s for s in the preimage $\tilde{S}_{aff}$ of S_{aff} and all τ_t for $t \in \mathrm{T}^0/\mathrm{T}^1$.

There is an involutive automorphism defined on $\tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$ by ([31, Corollary 2]):

$$\iota : \tau_w \mapsto (-q)^{\ell(w)} \tau_{w^{-1}}^{-1}. \tag{1.11}$$

and it actually yields an involution on $\tilde{\mathrm{H}}_{\mathbb{Z}}$. Inflate the character $\epsilon_C : \mathrm{W} \to \{\pm 1\}$ defined in 1.2.2 to a character of $\tilde{\mathrm{W}}$. We define the following $\mathbb{Z}$-linear involution υ_C of $\tilde{\mathrm{H}}_{\mathbb{Z}}$ by:

$$\upsilon_C(\tau_w) = \epsilon_C(w)\tau_w \qquad \text{for any } w \in \tilde{\mathrm{W}}.$$

It is the identity on the affine subring $\tilde{\mathrm{H}}_{\mathbb{Z}}^{aff}$. We will consider the following $\mathbb{Z}$-linear involution on $\tilde{\mathrm{H}}_{\mathbb{Z}}$:

$$\iota_C = \iota \circ \upsilon_C. \tag{1.12}$$

Remark 1.7. The involution ι_C fixes all τ_w for $w \in \tilde{\mathrm{W}}$ with length zero.

1.2.8. *Idempotents.* Let R be a ring containing an inverse for $(q1_{\mathrm{R}} - 1)$ and a primitive $(q-1)^{\mathrm{th}}$ root of 1_{R}. The group of characters of $\mathrm{T}^0/\mathrm{T}^1 \simeq \overline{\mathbf{T}}(\mathbb{F}_q)$ with values in $\mathrm{R}^\times$ is isomorphic to the group of characters of $\overline{\mathbf{T}}(\mathbb{F}_q)$ with values in $\mathbb{F}_q^\times$ which we denote by $\hat{\overline{\mathbf{T}}}(\mathbb{F}_q)$. To $\xi \in \hat{\overline{\mathbf{T}}}(\mathbb{F}_q)$ we attach the idempotent element $\epsilon_\xi \in \tilde{\mathrm{H}}_{\mathrm{R}}$ as in [31] (definition recalled in [24, 2.3.4]). The sum of all ϵ_ξ is the identity in $\tilde{\mathrm{H}}_{\mathrm{R}}$.

The field k is an example of ring R as above. We now turn to the quadratic relations in $\tilde{\mathrm{H}}_k$. For $A \in \Pi_{aff}$, choose the lift $n_A \in \mathrm{G}$ for s_A defined after fixing an épinglage for G as in [31, 1.2] (recalled in [24, 2.2.5]). For $\xi \in \hat{\overline{\mathbf{T}}}(\mathbb{F}_q)$, we have in $\tilde{\mathrm{H}}_k$:

$$\begin{cases} \text{if } s_A.\xi = \xi \text{ then } \epsilon_\xi \tau_{n_A}^2 = -\epsilon_\xi \tau_{n_A} \\ \text{if } s_A.\xi \neq \xi \text{ then } \epsilon_\xi \tau_{n_A}^2 = 0. \end{cases} \tag{1.13}$$

Remark 1.8. In $\tilde{\mathrm{H}}_k$ we have $\tau_{n_A}\,\iota(\tau_{n_A}) = 0$ for all $A \in S_{aff}$. Furthermore, $\iota(\tau_{n_A}) + \tau_{n_A}$ lies in the subalgebra of $\tilde{\mathrm{H}}_k$ generated by all τ_t, $t \in \mathrm{T}^0/\mathrm{T}^1$, or equivalently by all ϵ_ξ, $\xi \in \overline{\mathbf{T}}(\mathbb{F}_q)$ (see [24, Remark 2.10] for an explicit formula).

1.2.9. *Parametrization of the weights.* The functions in $\tilde{\mathrm{H}}_{\mathbb{Z}}$ with support in K form a subring $\tilde{\mathfrak{H}}_{\mathbb{Z}}$. It has $\mathbb{Z}$-basis the set of all τ_w for $w \in \tilde{\mathfrak{W}}$. Denote by $\tilde{\mathfrak{H}}_k$ the k-algebra $\tilde{\mathfrak{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$. The simple modules of $\tilde{\mathfrak{H}}_k$ are one dimensional [28, (2.11)].

An irreducible smooth k-representation of K is called a weight. By [9, Corollary 7.5] the weights are in one-to-one correspondence with the characters of $\tilde{\mathfrak{H}}_k$ *via* $\rho \to \rho^{\tilde{\mathrm{I}}}$. To a character $\chi : \tilde{\mathfrak{H}}_k \to k$ is attached the morphism $\bar{\chi} : \mathrm{T}^0/\mathrm{T}^1 \to k^\times$ such that $\bar{\chi}(t) = \chi(\tau_t)$ for all $t \in \mathrm{T}^0/\mathrm{T}^1$

and the set $\Pi_{\bar\chi}$ of all simple roots $\alpha \in \Pi$ such that $\bar\chi$ is fixed by s_α. We then have $\chi(\tau_{\tilde s_\alpha}) = 0$ for all $\alpha \in \Pi - \Pi_{\bar\chi}$, where $\tilde s_\alpha \in \tilde{\mathrm{W}}$ is any lift for $s_\alpha \in \mathrm{W}$. We denote by Π_χ the subset of all $\alpha \in \Pi_{\bar\chi}$ such that $\chi(\tau_{\tilde s_\alpha}) = 0$. The character χ is determined by the data of $\bar\chi$ and Π_χ (see also [24, 3.4]).

Remark 1.9. Choosing a standard facet F is equivalent to choosing the subset Π_F of Π of the simple roots taking value zero on F. The standard facet corresponding to Π_χ in the previous discussion will be denoted by F_χ.

2. On the center of the pro-p Iwahori Hecke algebra in characteristic p

2.1. Commutative subrings of the generic pro-p Hecke ring. Let σ be a sign and F a standard facet.

As in [24, 4.1.1], we introduce the multiplicative injective map

$$\Theta_F^\sigma : \tilde{\mathrm{X}}_*(\mathrm{T}) \longrightarrow \tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$$

and the elements $\mathcal{B}_F^\sigma(\lambda) := q^{\ell(e^\lambda)/2}\Theta_F^\sigma(\lambda)$ for all $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$. Recall that $\mathcal{B}_F^\sigma(\lambda) = \tau_{e^\lambda}$ if $\lambda \in \mathscr{C}^\sigma(F)$ (and this remark is sufficient to retrieve the formula for $\mathcal{B}_F^\sigma(\lambda)$ for a general λ). The map $\mathcal{B}_F^\sigma$ does not respect the product in general, but it is multiplicative when restricted to any Weyl chamber (see [24, Remark 4.3]).

For any coweight $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$, the element $\mathcal{B}_F^\sigma(\lambda)$ lies in $\tilde{\mathrm{H}}_{\mathbb{Z}}$. Furthermore combining Lemma 1.2(2) and [24, Lemmas 2.11 and 4.4]:

$$\iota_C(\mathcal{B}_F^+(\lambda)) = \mathcal{B}_F^-(\lambda). \tag{2.1}$$

We consider the commutative subring $\mathcal{A}_F^\sigma := \tilde{\mathrm{H}}_{\mathbb{Z}} \cap \mathrm{Im}(\Theta_F^\sigma)$. By [24, Prop. 4.5], it is a free $\mathbb{Z}$-module with basis the set of all $\mathcal{B}_F^\sigma(\lambda)$ for $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$. Since the Weyl chambers (in $\tilde{\mathrm{X}}_*(\mathrm{T})$) are finitely generated semigroups, $\mathcal{A}_F^\sigma$ is finitely generated as a ring.

Remark 2.1. Note that $\mathcal{B}_C^+ = \mathcal{B}_{x_0}^-$ (resp. $\mathcal{B}_C^- = \mathcal{B}_{x_0}^+$) coincides with the integral Bernstein map E^+ (resp. E) introduced in [31] and $\mathcal{A}_C^+$ (resp. $\mathcal{A}_C^-$) with the commutative ring denoted by $\mathcal{A}^{+,(1)}$ (resp. $\mathcal{A}^{(1)}$) in [31, Theorem 2].

The following is a direct consequence of the "fundamental lemma" proved in [13, §5] and adapted to the pro-p-Iwahori Hecke algebra in [31, Lemma 13].

Lemma 2.2. *Let F be a standard facet and σ a sign. For any $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$, we have*

$$\mathcal{B}_F^\sigma(\lambda) = \tau_{e^\lambda} + \sum_{w < e^\lambda} a_w \tau_w$$

where $a_w \in \mathbb{Z}$ and w ranges over the set of $w \in \tilde{\mathrm{W}}$ such that $w < e^\lambda$. For those w, we have in particular $\ell(w) < \ell(e^\lambda)$.

Identify $\mathrm{X}_*(\mathrm{T})$ with its image in $\tilde{\mathrm{X}}_*(\mathrm{T})$ *via* (1.8). We denote by $(\mathcal{A}_F^\sigma)^\circ$ the intersection

$$(\mathcal{A}_F^\sigma)^\circ := \tilde{\mathrm{H}}_{\mathbb{Z}} \cap \Theta_F^\sigma(\mathrm{X}_*(\mathrm{T})) \subseteq \mathcal{A}_F^\sigma.$$

A $\mathbb{Z}$-basis for $(\mathcal{A}_F^\sigma)^\circ$ is given by all $\mathcal{B}_F^\sigma(\lambda)$ for $\lambda \in \mathrm{X}_*(\mathrm{T})$. It is finitely generated as a ring.

Proposition 2.3. *The commutative $\mathbb{Z}$-algebra $\mathcal{A}_F^\sigma$ is isomorphic to the tensor product of the $\mathbb{Z}$-algebras $\mathbb{Z}[\mathrm{T}^0/\mathrm{T}^1]$ and $(\mathcal{A}_F^\sigma)^\circ$.*

Proof. Since the exact sequence (1.9) splits, $\mathcal{A}_F^\sigma$ is a free $(\mathcal{A}_F^\sigma)^\circ$-module with basis the set of all τ_t for $t \in \mathrm{T}^0/\mathrm{T}^1$. Recall indeed that $\mathcal{B}_F^\sigma(\lambda + t) = \mathcal{B}_F^+(\lambda)\tau_t = \tau_t\mathcal{B}_F^\sigma(\lambda)$ for all $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$ and $t \in \mathrm{T}^0/\mathrm{T}^1$. □

2.2. **On the center of the generic pro-p Hecke ring.** Recall the following result proved in [31, Theorem 4]. The ring $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is finitely generated as a module over its center $\mathcal{Z}(\tilde{\mathrm{H}}_{\mathbb{Z}}) = (\mathcal{A}_C^+)^{\mathfrak{W}}$ and the latter has $\mathbb{Z}$-basis the set of all

$$\sum_{\lambda' \in \mathcal{O}} \mathcal{B}_C^+(\lambda') \tag{2.2}$$

where $\mathcal{O}$ ranges over the $\mathfrak{W}$-orbits in $\tilde{\mathrm{X}}_*(\mathrm{T})$. Moreover, $\mathcal{Z}(\tilde{\mathrm{H}}_{\mathbb{Z}})$ is a finitely generated $\mathbb{Z}$-algebra. One can also find a proof of those results in [26] where abstract pro-p-Hecke algebras are introduced. This work was brought to my attention by P. Schneider.

2.2.1. We denote by $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathbb{Z}})$ the intersection of $(\mathcal{A}_C^+)^\circ$ with $\mathcal{Z}(\tilde{\mathrm{H}}_{\mathbb{Z}})$. It has $\mathbb{Z}$-basis the set of all

$$z_\lambda := \sum_{\lambda' \in \mathcal{O}(\lambda)} \mathcal{B}_C^+(\lambda') \text{ for } \lambda \in \mathrm{X}_*^+(\mathrm{T}) \tag{2.3}$$

where we denote by $\mathcal{O}(\lambda)$ the $\mathfrak{W}$-orbit of λ.

Proposition 2.4. (1) *The left (resp. right) $(\mathcal{A}_C^+)^\circ$-module $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is finitely generated.*
(2) *As a $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathbb{Z}})$-module, $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is finitely generated.*
(3) *$\mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathbb{Z}})$ is a finitely generated $\mathbb{Z}$-algebra.*

Proof. Using Proposition 2.3 and [31, Theorems 3 and 4] which state that $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is finitely generated over $\mathcal{A}_C^+$ (see Remark 2.1), we see that $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is finitely generated over $(\mathcal{A}_C^+)^\circ$. The other statements follow from [3, §1 n. 9 Thm 2] because $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathbb{Z}})$ is the ring of $\mathfrak{W}$-invariants of $(\mathcal{A}_C^+)^\circ$ and $\mathbb{Z}$ is noetherian. □

2.2.2. Given a ring R, we denote by $\tilde{\mathrm{H}}_{\mathrm{R}}$ the R-algebra $\tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{R}$: we identify q with its image in R. We denote by $\mathcal{Z}(\tilde{\mathrm{H}}_{\mathrm{R}})$ (resp. $(\mathcal{A}_C^+)_R$, $(\mathcal{A}_C^+)_R^\circ$ and $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathrm{R}})$) the image of $\mathcal{Z}(\tilde{\mathrm{H}}_{\mathbb{Z}})$ (resp. $\mathcal{A}_C^+$, $(\mathcal{A}_C^+)^\circ$ and $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathbb{Z}})$) in $\tilde{\mathrm{H}}_{\mathrm{R}}$. By the work of [26], $\mathcal{Z}(\tilde{\mathrm{H}}_{\mathrm{R}})$ is not only contained in but equal to the center of $\tilde{\mathrm{H}}_{\mathrm{R}}$.

For $\lambda \in \mathrm{X}_*(\mathrm{T})$ and $w \in \tilde{\mathrm{W}}$, we denote the elements $\mathcal{B}_F^{\sigma}(\lambda)$ (resp. τ_w) and their respective images in $\tilde{\mathrm{H}}_{\mathrm{R}}$ identically. An R-basis for $\mathcal{Z}^{\circ}(\tilde{\mathrm{H}}_{\mathrm{R}})$ is given by the set of all z_λ for $\lambda \in \mathrm{X}_*^+(\mathrm{T})$, where again we identify the element z_λ with its image in $\tilde{\mathrm{H}}_{\mathrm{R}}$.

From Proposition 2.4 we deduce:

Proposition 2.5. *Let* R *be a field. A morphism of* R*-algebras* $\mathcal{Z}^{\circ}(\tilde{\mathrm{H}}_{\mathrm{R}}) \to \mathrm{R}$ *can be extended to a morphism of* R*-algebras* $\mathcal{Z}(\tilde{\mathrm{H}}_{\mathrm{R}}) \to \mathrm{R}$.

2.2.3. The ring $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is only determined by the choice of the chamber C. In the process of constructing $\mathcal{Z}^{\circ}(\tilde{\mathrm{H}}_{\mathbb{Z}})$, we first fixed a hyperspecial vertex x_0 of C and then an apartment $\mathscr{A}$ containing C.

Proposition 2.6. *The construction of the ring* $\mathcal{Z}^{\circ}(\tilde{\mathrm{H}}_{\mathbb{Z}})$ *is not affected by*

- *the choice of another apartment* $\mathscr{A}'$ *containing* C*.*
- *the choice of another vertex* x_0' *of* C *provided it is* G*-conjugate to* x_0*.*

Proof. Let g be in the stabilizer $\mathcal{P}_C^{\dagger}$ of C in G. Let $\mathrm{T}' := g\mathrm{T}g^{-1}$ and $x_0' = gx_0g^{-1}$. The apartment $\mathscr{A}'$ corresponding to T' contains C and x_0' is a hyperspecial vertex of C. Starting from T' and x_0' we proceed to the construction of the corresponding commutative subring $\mathcal{Z}^{\circ}(\tilde{\mathrm{H}}_{\mathbb{Z}})'$ of the center of $\tilde{\mathrm{H}}_{\mathbb{Z}}$. Since $g \in \mathcal{P}_C^{\dagger}$, we have $\tilde{\mathrm{I}}g\tilde{\mathrm{I}} = \tilde{\mathrm{I}}\hat{\omega}\tilde{\mathrm{I}} = \tilde{\mathrm{I}}\hat{\omega}$ for some $\omega \in \tilde{\Omega}$. Since this element ω has length zero, for $\lambda \in \mathrm{X}_*(\mathrm{T})$ the characteristic function of $\tilde{\mathrm{I}}g\lambda(\varpi)g^{-1}\tilde{\mathrm{I}}$ is equal to the product $\tau_g\tau_{\lambda(\varpi)}\tau_g^{-1}$. Therefore, the restriction to $\mathrm{X}_*(\mathrm{T})$ of the new map $(\mathcal{B}_C^+)'$ corresponding to the choice of x_0' and T' is

$$\mathbb{Z}[\mathrm{X}_*(\mathrm{T}')] \longrightarrow \tilde{\mathrm{H}}_{\mathbb{Z}},\ \lambda \mapsto \tau_g\mathcal{B}_C^+(g^{-1}\lambda g)\tau_g^{-1}.$$

The element $z_\lambda' \in \mathcal{Z}^{\circ}(\tilde{\mathrm{H}}_{\mathbb{Z}})'$ corresponding to the choice of $\lambda \in \mathrm{X}_*^+(\mathrm{T}') = g\mathrm{X}_*^+(\mathrm{T})g^{-1}$ is therefore $\tau_g z_{g^{-1}\lambda g}\tau_g^{-1} = z_{g^{-1}\lambda g}$. We have proved that $\mathcal{Z}^{\circ}(\tilde{\mathrm{H}}_{\mathbb{Z}})' = \mathcal{Z}^{\circ}(\tilde{\mathrm{H}}_{\mathbb{Z}})$.

By Proposition 1.4 i and Remark 1.5

- changing $\mathscr{A}$ into another apartment $\mathscr{A}'$ containing C and
- changing x_0 into another vertex x_0' of C which is G-conjugate to x_0

can be made independently of each other by conjugating by an element of I and of $\mathcal{P}_C^{\dagger} \cap N_{\mathrm{G}}(\mathrm{T})$ respectively. We have checked that these changes do not affect $\mathcal{Z}^{\circ}(\tilde{\mathrm{H}}_{\mathbb{Z}})$.

∎

If **G** is of adjoint type or $\mathbf{G} = \mathrm{GL}_n$, then all hyperspecial vertices are conjugate ([29, 2.5]):

Corollary 2.7. *If* **G** *is of adjoint type or* $\mathbf{G} = \mathrm{GL}_n$*, then* $\mathcal{Z}^{\circ}(\tilde{\mathrm{H}}_{\mathbb{Z}})$ *depends only on the choice of* ϖ*.*

2.3. **An affine semigroup algebra in the center of the pro-p Iwahori Hecke algebra in characteristic p.** We will use the following observation several times in this subsection. Let F be a standard facet and σ a sign. For $\mu_1, \mu_2 \in \mathrm{X}_*(\mathrm{T})$, we have in $\tilde{\mathrm{H}}_k$:

$$\mathcal{B}_F^\sigma(\mu_1)\mathcal{B}_F^\sigma(\mu_2) = \begin{cases} \mathcal{B}_F^\sigma(\mu_1+\mu_2) & \text{if } \mu_1 \text{ and } \mu_2 \text{ lie in a common Weyl chamber} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

In $\tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$ we have indeed $\mathcal{B}_F^\sigma(\mu_1)\mathcal{B}_F^\sigma(\mu_2) = q^{(\ell(e^{\mu_1})+\ell(e^{\mu_2})-\ell(e^{\mu_1+\mu_2}))/2}\mathcal{B}_F^\sigma(\mu_1+\mu_2)$. If μ_1 and μ_2 lie in a common Weyl chamber, then $\ell(e^{\mu_1})+\ell(e^{\mu_2})-\ell(e^{\mu_1+\mu_2})$ is zero; otherwise, there is $\alpha \in \Pi$ satisfying $\langle \mu_1, \alpha\rangle\langle\mu_2,\alpha\rangle < 0$ which implies that this quantity is ≥ 2. Projecting in $\tilde{\mathrm{H}}_k$, we obtain the result.

2.3.1. *The structure of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$.*

Proposition 2.8. *The map*

$$\begin{array}{ccc} k[\mathrm{X}_*^+(\mathrm{T})] & \longrightarrow & \mathcal{Z}^\circ(\tilde{\mathrm{H}}_k) \\ \lambda & \longmapsto & z_\lambda \end{array} \tag{2.5}$$

is an isomorphism of k-algebras.

Proof. We already know that (2.7) maps a k-basis for $k[\mathrm{X}_*^+(\mathrm{T})]$ onto a k-basis for $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$. We have to check that it respects the product. Let $\lambda_1, \lambda_2 \in \mathrm{X}_*^+(\mathrm{T})$ with respective $\mathfrak{W}$-orbits $\mathcal{O}(\lambda_1)$ and $\mathcal{O}(\lambda_2)$. We consider the product

$$z_{\lambda_1}z_{\lambda_2} = \sum_{\mu_1 \in \mathcal{O}(\lambda_1),\, \mu_2 \in \mathcal{O}(\lambda_2)} \mathcal{B}_F^\sigma(\mu_1)\mathcal{B}_F^\sigma(\mu_2) \in \tilde{\mathrm{H}}_k.$$

A Weyl chamber in $\mathrm{X}_*(\mathrm{T})$ is a $\mathfrak{W}$-conjugate of $\mathrm{X}_*^+(\mathrm{T})$. Given a Weyl chamber and a coweight (in $\mathrm{X}_*(\mathrm{T})$), there is a unique $\mathfrak{W}$-conjugate of the coweight in the chosen Weyl chamber. The map $(\mu_1,\mu_2) \mapsto \mu_1+\mu_2$ yields a bijection between the set of all $(\mu_1,\mu_2) \in \mathcal{O}(\lambda_1)\times\mathcal{O}(\lambda_2)$ such that μ_1 and μ_2 lie in the same Weyl chamber and the $\mathfrak{W}$-orbit $\mathcal{O}(\lambda_1+\lambda_2)$ of $\lambda_1+\lambda_2$: it is indeed surjective and one checks that the two sets in question have the same size using that the stabilizer in $\mathfrak{W}$ of $\lambda_1+\lambda_2$ is the intersection of stabilizer in $\mathfrak{W}$ of λ_1 with the stabilizer of λ_2. Together with (2.4), this proves that $z_{\lambda_1+\lambda_2} = z_{\lambda_1}z_{\lambda_2}$. □

2.3.2. Since $\mathrm{X}_*(\mathrm{T})$ is a free abelian group (of rank $dim(\mathrm{T})$), the k-algebra $k[\mathrm{X}_*(\mathrm{T})]$ is isomorphic to an algebra of Laurent polynomials and has a trivial nilradical. By Gordan's Lemma ([19, p. 7]), $\mathrm{X}_*^+(\mathrm{T})$ is finitely generated as a semigroup. So $k[\mathrm{X}_*^+(\mathrm{T})]$ is a finitely generated k-algebra and its Jacobson radical coincides with its nilradical (see for example [Milne, A primer of commutative algebra, Prop 11.8]). The Jacobson radical of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ is therefore trivial.

Proposition 2.9. *The Jacobson radical of $\mathcal{Z}(\tilde{\mathrm{H}}_k)$ is trivial.*

Proof. Since $\mathcal{Z}(\tilde{\mathrm{H}}_k)$ is a finitely generated k-algebra contained in $(\mathcal{A}_C^+)_k$, it is enough to prove that the nilradical of $(\mathcal{A}_C^+)_k$ is trivial. By Proposition 2.3, the k-algebra $(\mathcal{A}_C^+)_k$ is isomorphic to the tensor product of $k[\mathrm{T}^0/\mathrm{T}^1]$ by $(\mathcal{A}_C^+)_k^\circ$ and since k has characteristic p, the k-algebra $k[\mathrm{T}^0/\mathrm{T}^1]$ is semisimple. Therefore, it remains to prove that the nilradical of $(\mathcal{A}_C^+)_k^\circ$ is trivial.

By definition (see the conventions in 2.2.2), the image of the k-linear injective map

$$\mathcal{B}_C^+ : k[\mathrm{X}_*(\mathrm{T})] \longrightarrow \tilde{\mathrm{H}}_k$$

coincides with $(\mathcal{A}_C^+)_k^\circ$.

Fact i. *Let $\lambda_0 \in \mathrm{X}_*^+(\mathrm{T})$ be a strongly dominant coweight. The ideal of $(\mathcal{A}_C^+)_k^\circ$ generated by $\mathcal{B}_C^+(\lambda_0)$ coincides with the image of the k-linear injective map*

$$\mathcal{B}_C^+ : k[\mathrm{X}_*^+(\mathrm{T})] \longrightarrow \tilde{\mathrm{H}}_k, \lambda \mapsto \mathcal{B}_C^+(\lambda + \lambda_0).$$

Its only nilpotent element is zero.

A generic element $a \in (\mathcal{A}_C^+)_k^\circ$ is a k-linear combination of elements $\mathcal{B}_C^+(\lambda)$ for $\lambda \in \mathrm{X}_*(\mathrm{T})$ and we say that $\lambda \in \mathrm{X}_*(\mathrm{T})$ is in the support of a if the coefficient of $\mathcal{B}_C^+(\lambda)$ is nonzero. Suppose that a is nilpotent and nontrivial. After conjugating by an element of $\mathfrak{W}$, we can suppose that there is an element in $\mathrm{X}_*^+(\mathrm{T})$ in the support of a. Then let $\lambda_0 \in \mathrm{X}_*^+(\mathrm{T})$ be a strongly dominant element. The element $a\mathcal{B}_C^+(\lambda_0)$ is nilpotent and by (2.4) it is nontrivial. By Fact i, we have a contradiction.

Proof of the fact: The first statement comes from (2.4). The restriction of $\mathcal{B}_C^+$ to $k[\mathrm{X}_*^+(\mathrm{T})]$ being an injective morphism of algebras, its image does not contain any nontrivial nilpotent element. It proves the fact.

□

Since k is algebraically closed, we have:

Corollary 2.10. *Let $z \in \mathcal{Z}(\tilde{\mathrm{H}}_k)$. If $\zeta(z) = 0$ for all characters $\zeta : \mathcal{Z}(\tilde{\mathrm{H}}_k) \to k$, then $z = 0$.*

2.3.3. *The center of the Iwahori-Hecke k-algebra.* Let R be a ring containing an inverse for $(q1_{\mathrm{R}} - 1)$ and a primitive $(q-1)^{\mathrm{th}}$ root of 1_{R}. We can apply the observations of 1.2.8 and consider the algebra

$$\tilde{\mathrm{H}}_{\mathrm{R}}(\xi) := \epsilon_\xi \tilde{\mathrm{H}}_{\mathrm{R}} \epsilon_\xi.$$

It can be seen as the algebra $\mathcal{H}(\mathrm{G}, \mathrm{I}, \xi^{-1})$ of G-endomorphisms of the representation $\epsilon_\xi \mathrm{ind}_{\mathrm{I}}^{\mathrm{G}} \mathbf{1}_{\mathrm{R}}$ which is isomorphic to the compact induction $\mathrm{ind}_{\mathrm{I}}^{\mathrm{G}} \xi^{-1}$ of ξ^{-1} seen as a R-character of I trivial on $\tilde{\mathrm{I}}$: denote by $1_{\mathrm{I},\xi^{-1}} \in \mathrm{ind}_{\mathrm{I}}^{\mathrm{G}} \xi^{-1}$ the unique fonction with support in I and value 1_{R} at 1_{G}, then the map

$$\tilde{\mathrm{H}}_{\mathrm{R}}(\xi) \to \mathcal{H}(\mathrm{G}, \mathrm{I}, \xi^{-1}),\ h \mapsto [1_{\mathrm{I},\xi^{-1}} \mapsto 1_{\mathrm{I},\xi^{-1}} h] \tag{2.6}$$

gives the identification. In particular, when $\xi = \mathbf{1}$ is the trivial character, then the algebra $\tilde{\mathrm{H}}_{\mathrm{R}}(\mathbf{1})$ identifies with the usual (and most studied) Iwahori-Hecke algebra $\mathrm{H}_{\mathrm{R}} = \mathrm{R}[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ with coefficients in R.

Remark 2.11. Let $\xi \in \hat{\bar{\mathbf{T}}}(\mathbb{F}_q)$. We have inclusions

$$\epsilon_\xi \mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathrm{R}}) \subseteq \epsilon_\xi \mathcal{Z}(\tilde{\mathrm{H}}_{\mathrm{R}}) \subseteq \mathcal{Z}(\tilde{\mathrm{H}}_{\mathrm{R}}(\xi))$$

where the latter space is the center of $\tilde{\mathrm{H}}_{\mathrm{R}}(\xi)$. The inclusion $\epsilon_\xi \mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathrm{R}}) \subseteq \mathcal{Z}(\tilde{\mathrm{H}}_{\mathrm{R}}(\epsilon_\xi))$ is strict in general. Choose ξ with trivial stabilizer in $\mathfrak{W}$. Then one easily checks that $\tilde{\mathrm{H}}_{\mathrm{R}}(\xi)$ is commutative with an R-basis indexed by the elements in $\mathrm{X}_*(\mathrm{T})$ (see the example of $\mathrm{GL}_2(\mathfrak{F})$ and $\mathrm{R} = k$ in [2, Proposition 13]) whereas a R-basis for $\epsilon_\xi \mathcal{Z}^\circ(\tilde{\mathrm{H}}_{\mathrm{R}})$ is indexed by $\mathrm{X}_*^+(\mathrm{T})$ by Proposition 2.8.

If $\xi = \mathbf{1}$ however, and more generally if ξ is fixed by all elements in $\mathfrak{W}$, these inclusions are equalities: one easily checks by direct comparison of the basis elements (2.2) and (2.3) that the first inclusion is an equality. The second one comes from the fact that ϵ_ξ is a central idempotent in $\tilde{\mathrm{H}}_{\mathrm{R}}$. In particular we have:

Theorem 2.12. *The center of the Iwahori-Hecke k-algebra $k[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ is isomorphic to $k[\mathrm{X}_*^+(\mathrm{T})]$.*

Proof. The map

$$\begin{array}{ccc} k[\mathrm{X}_*^+(\mathrm{T})] & \longrightarrow & \epsilon_{\mathbf{1}} \mathcal{Z}(\tilde{\mathrm{H}}_k) \\ \lambda & \longmapsto & \epsilon_{\mathbf{1}} z_\lambda \end{array} \tag{2.7}$$

is surjective by the previous discussion. It is easily checked to be injective using Lemma 2.2. Compare with [30, (1.6.5)]. □

3. The central Bernstein functions in the pro-p Iwahori-Hecke ring

Let $\mathcal{O}$ be a $\mathfrak{W}$-orbit in $\tilde{\mathrm{X}}_*(\mathrm{T})$. We call the central element of $\tilde{\mathrm{H}}_{\mathbb{Z}}$

$$z_{\mathcal{O}} := \sum_{\lambda' \in \mathcal{O}} \mathcal{B}_C^+(\lambda') \tag{2.2}$$

the associated central Bernstein function.

3.1. **The support of the central Bernstein functions.** For $h \in \tilde{\mathrm{H}}_{\mathbb{Z}}$, the set of all $w \in \tilde{\mathrm{W}}$ such that $h(\hat{w}) \neq 0$ is called the support *support* of h. For $\mathcal{O}$ a $\mathfrak{W}$-orbit in $\tilde{\mathrm{X}}_*(\mathrm{T})$ we denote by $\ell_{\mathcal{O}}$ the common length of all the coweights in $\mathcal{O}$.

Lemma 3.1. *Let $\mathcal{O}$ be a $\mathfrak{W}$-orbit in $\tilde{\mathrm{X}}_*(\mathrm{T})$. The support of $z_{\mathcal{O}}$ (resp. $\iota_C(z_{\mathcal{O}})$) contains the set of all e^μ for $\mu \in \mathcal{O}$. Any other element in the support of $z_{\mathcal{O}}$ (resp. $\iota_C(z_{\mathcal{O}})$) has length $< \ell_{\mathcal{O}}$.*

Proof. This is a consequence of Lemma 2.2 (and (2.1)). □

Proposition 3.2. *The involution ι_C fixes the elements in the center $\mathcal{Z}(\tilde{\mathrm{H}}_{\mathbb{Z}})$ of $\tilde{\mathrm{H}}_{\mathbb{Z}}$.*
In particular, for $\mathcal{O}$ a $\mathfrak{W}$-orbit in $\tilde{\mathrm{X}}_*(\mathrm{T})$, the element $\sum_{\lambda'\in\mathcal{O}}\mathcal{B}^{\sigma}_C(\lambda')\in\tilde{\mathrm{H}}_{\mathbb{Z}}$ does not depend on the sign σ.

Proof. A $\mathbb{Z}$-basis for $\mathcal{Z}(\tilde{\mathrm{H}}_{\mathbb{Z}})$ is given by the central Bernstein functions $z_{\mathcal{O}}$ where $\mathcal{O}$ ranges over the $\mathfrak{W}$-orbits in $\tilde{\mathrm{X}}_*(\mathrm{T})$. We prove that ι_C fixes $z_{\mathcal{O}}$ by induction on $\ell_{\mathcal{O}}$.

If $\ell_{\mathcal{O}}=0$, then we conclude with Remark 1.7. Let $\mathcal{O}$ a $\mathfrak{W}$-orbit in $\tilde{\mathrm{X}}_*(\mathrm{T})$ such that $\ell_{\mathcal{O}}>0$. The element $\iota_C(z_{\mathcal{O}})$ is central in $\tilde{\mathrm{H}}_{\mathbb{Z}}$. By the previous lemma one easily sees that that $\iota_C(z_{\mathcal{O}})$ decomposes as a sum

$$\iota_C(z_{\mathcal{O}})=z_{\mathcal{O}}+\sum_{\mathcal{O}'}a_{\mathcal{O}'}z_{\mathcal{O}'}$$

where $\mathcal{O}'$ ranges over a finite set of $\mathfrak{W}$-orbits in $\tilde{\mathrm{X}}_*(\mathrm{T})$ such that $\ell_{\mathcal{O}'}<\ell_{\mathcal{O}}$. By induction hypothesis and applying the involution ι_C we get

$$z_{\mathcal{O}}=\iota_C(z_{\mathcal{O}})+\sum_{\mathcal{O}'}a_{\mathcal{O}'}z_{\mathcal{O}'}$$

so that $2(\iota(z_{\mathcal{O}})-z_{\mathcal{O}}))=0$. Since $\tilde{\mathrm{H}}_{\mathbb{Z}}$ has no $\mathbb{Z}$-torsion, $\iota(z_{\mathcal{O}})=z_{\mathcal{O}}$. The second statement follows from (2.1).

$\square$

When G is semisimple, the projection in $\tilde{\mathrm{H}}_k$ of the equality proved in Proposition 3.2 can be obtained independently using the duality for finite length $\tilde{\mathrm{H}}_k$-modules defined in [23]:

Proposition 3.3. *Suppose that* G *is semisimple. The element $\sum_{\lambda'\in\mathcal{O}}\mathcal{B}^{\sigma}_C(\lambda')\in\tilde{\mathrm{H}}_k$ is fixed by the involution ι_C and therefore does not depend on the sign σ.*

Proof. We suppose that G is semisimple. Let $\mathcal{O}$ be a $\mathfrak{W}$-orbit in $\tilde{\mathrm{X}}_*(\mathrm{T})$. We want to prove that in $\tilde{\mathrm{H}}_k$ we have $z_{\mathcal{O}}=\iota_C(z_{\mathcal{O}})$.
Let $\zeta:\mathcal{Z}(\tilde{\mathrm{H}}_k)\to k$ a character and $M=\tilde{\mathrm{H}}_k\otimes_{\mathcal{Z}(\tilde{\mathrm{H}}_k)}\zeta$ the induced $\tilde{\mathrm{H}}_k$-module. It is a finite dimensional k-module and therefore, by [23, Corollary 6.12] we have and isomorphism of right $\tilde{\mathrm{H}}_k$-modules

$$\mathrm{Ext}^d_{\tilde{\mathrm{H}}_k}(M,\tilde{\mathrm{H}}_k)=\mathrm{Hom}_k(\iota_C^*M,k)$$

where d is the semisimple rank of G and ι_C^*M denotes the left $\tilde{\mathrm{H}}_k$-module M with action twisted by the involution defined by (1.12). The category of left $\tilde{\mathrm{H}}_k$-modules is naturally a $\mathcal{Z}(\tilde{\mathrm{H}}_k)$-linear category so that for X and Y two given left $\tilde{\mathrm{H}}_k$-modules, $\mathrm{Ext}^d(X,Y)$ inherits a structure of central $\mathcal{Z}(\tilde{\mathrm{H}}_k)$-bimodule. The right $\tilde{\mathrm{H}}_k$-module $\mathrm{Ext}^d_{\tilde{\mathrm{H}}_k}(M,\tilde{\mathrm{H}}_k)$ therefore has a central character equal to ζ. On the other hand, $\mathrm{Hom}_k(\iota_C^*M,k)$ has $\zeta\circ\iota_C$ as a central character. Therefore, $\zeta(z_{\mathcal{O}})=\zeta\circ\iota_C(z_{\mathcal{O}})$. By Corollary 2.10, we have the required equality $z_{\mathcal{O}}=\iota_C(z_{\mathcal{O}})$. $\square$

3.2. **Statement of the "flower" lemma.** The following lemma will be proved in 3.3.3.

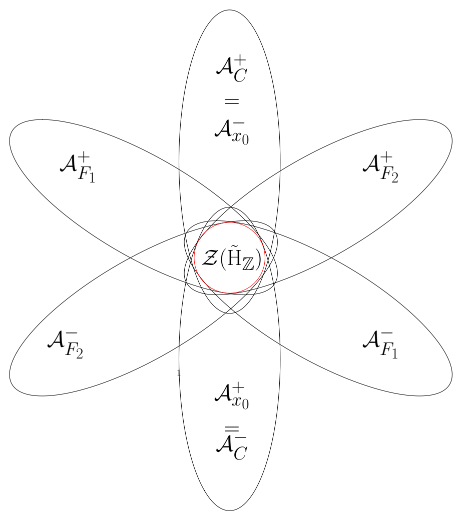
Lemma 3.4. *For $\mathcal{O}$ a $\mathfrak{W}$-orbit in $\tilde{\mathrm{X}}_*(\mathrm{T})$, the element*

$$\sum_{\lambda\in\mathcal{O}} \mathcal{B}^{\sigma}_{F}(\lambda)$$

in $\tilde{\mathrm{H}}_{\mathbb{Z}}$ does not depend on the choice of the standard facet F and of the sign σ.

Corollary 3.5. *The center of $\tilde{\mathrm{H}}_{\mathbb{Z}}$ is contained in the intersection of all the commutative rings $\mathcal{A}^{\sigma}_{F}$ for F a standard facet and σ a sign.*

In the case of $\mathrm{G} = \mathrm{GL}_3(\mathfrak{F})$, there are 4 standard facets: the vertex x_0, the chamber C and two edges F_1 and F_2. The corollary is illustrated by the diagram below.



3.3. **Inducing the generalized integral Bernstein functions.** We study the behavior of the integral Bernstein maps upon parabolic induction and then prove Lemma 3.4.

3.3.1. Consider a semistandard Levi subgroup of G. It corresponds as in [24, 2.3.3] to the choice of standard facet F and we denote it by M_F. The root datum attached to the choice of the split torus T in M_F is $(\Phi_F, \mathrm{X}^*(\mathrm{T}), \check{\Phi}_F, \mathrm{X}_*(\mathrm{T}))$ (notations in 1.2.4). The extended Weyl group of M_F is $\mathrm{W}_F = (N_{\mathrm{G}}(\mathrm{T})\cap \mathrm{M}_F)/\mathrm{T}^0$. It is isomorphic to the semidirect product $\mathfrak{W}_F \ltimes \mathrm{X}_*(\mathrm{T})$

where $\mathfrak{W}_F$ is the finite Weyl group $(N_{\mathrm{G}}(\mathrm{T})\cap \mathrm{M}_F)/\mathrm{T}$ (also defined in 1.2.4). We denote by ℓ_F its length function and by $\underset{F}{\leq}$ the Bruhat order on W_F.

Set $\tilde{\mathrm{W}}_F = (N_{\mathrm{G}}(\mathrm{T})\cap \mathrm{M}_F)/\mathrm{T}^1$. It is a subgroup of $\tilde{\mathrm{W}}$. The double cosets of M_F modulo its pro-p Iwahori subgroup $\tilde{\mathrm{I}}\cap \mathrm{M}_F$ are indexed by the elements in $\tilde{\mathrm{W}}_F$. For $w\in \mathrm{W}_F$, we denote by τ_w^F the characteristic function of the double coset containing the lift $\hat{w}$ for w (which lies in $\in N_{\mathrm{G}}(\mathrm{T})\cap \mathrm{M}_F$). The set of all $(\tau_w^F)_{w\in \mathrm{W}_F}$ is a basis for the pro-p Iwahori Hecke ring $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ of the $\mathbb{Z}$-valued functions with compact support in $(\tilde{\mathrm{I}}\cap \mathrm{M}_F)\backslash \mathrm{M}_F/(\tilde{\mathrm{I}}\cap \mathrm{M}_F)$. The ring $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ does not inject in $\tilde{\mathrm{H}}_{\mathbb{Z}}$ in general.

An element in $w\in \mathrm{W}_F$ is called F-positive if $w^{-1}(\Phi^+-\Phi_F^+)\subset \Phi_{aff}^+$. For example for $\lambda\in \mathrm{X}_*(\mathrm{T})$, the element e^λ is F-positive if $\langle\lambda,\alpha\rangle\geq 0$ for all $\alpha\in\Phi^+-\Phi_F^+$. In this case, we will say that the coweight λ itself is F-positive. If furthermore $\langle\lambda,\alpha\rangle>0$ for $\alpha\in\Phi^+-\Phi_F^+$ and $\langle\lambda,\alpha\rangle=0$ for $\alpha\in\Phi_F^+$, then it is called strongly F-positive. The F-positive coweights are the $\mathfrak{W}_F$-conjugates of the dominant coweights. An element in W_F is F-positive if and only if it belongs to $\mathfrak{W}_F e^\lambda \mathfrak{W}_F$ for some F-positive coweight $\lambda\in \mathrm{X}_*(\mathrm{T})$. If μ and $\nu\in \mathrm{X}_*(\mathrm{T})$ are F-positive coweights such that $\mu-\nu$ is also F-positive, then we have the equality (see [24, 1.2] for example)

$$\ell(e^{\mu-\nu})+\ell(e^\nu)-\ell(e^\mu)=\ell_F(e^{\mu-\nu})+\ell_F(e^\nu)-\ell_F(e^\mu) \tag{3.1}$$

An element in $\tilde{\mathrm{W}}_F$ will be called F-positive if its projection in W_F is F-positive.

The subspace of $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ generated over $\mathbb{Z}$ by all τ_w^F for F-positive $w\in\tilde{\mathrm{W}}_F$ is denoted by $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^+$. It is in fact a ring and there is an injection of rings

$$\begin{array}{rccc} j_F^+: & \tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^+ & \longrightarrow & \tilde{\mathrm{H}}_{\mathbb{Z}} \\ & \tau_w^F & \longmapsto & \tau_w \end{array}$$

which extends to an injection of $\mathbb{Z}[q^{\pm 1/2}]$-algebras

$$j_F:\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)\otimes_{\mathbb{Z}}\mathbb{Z}[q^{\pm1/2}]\to\tilde{\mathrm{H}}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Z}[q^{\pm1/2}].$$

This is a classical result for complex Hecke algebras ([7, (6.12)]). The argument is valid over $\mathbb{Z}[q^{\pm1/2}]$.

Remark 3.6. An element $w\in\tilde{\mathrm{W}}_F$ is called F-negative (resp. strongly F-negative) if w^{-1} is F-positive (resp. strongly F-positive) and as before, $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ contains as a subring the space $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^-$ generated over $\mathbb{Z}$ by all τ_w^F for F-negative $w\in\tilde{\mathrm{W}}_F$. There is an injection of rings $j_F^-:\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^-\longrightarrow\tilde{\mathrm{H}}_{\mathbb{Z}},\tau_w^F\longmapsto\tau_w$.

Fact ii. *Let $v\in\mathrm{W}_F$ such that $v\underset{F}{\leq}e^\lambda$ for $\lambda\in\mathrm{X}_*(\mathrm{T})$ a F-positive coweight. Then v is F-positive.*

Proof. Suppose first that λ is dominant. Then the claim is [24, Lemma 2.9.ii]. In general, λ is a $\mathfrak{W}_F$-conjugate of a dominant coweight λ_0: there is $u\in\mathfrak{W}_F$ such that $e^\lambda=ue^{\lambda_0}u^{-1}$. We argue

by induction on $\ell_F(u)$. Let s be a simple reflection in $\mathfrak{W}_F$ such that $\ell_F(su) = \ell_F(u) - 1$. By the properties of Bruhat order (see [13, Lemma 4.3] for example), one of v, vs, sv, svs is $\underset{F}{\leq} se^{\lambda}s$ and by induction this element is F-positive, which implies that v is a F-positive.

□

3.3.2. Let $F' \subseteq \overline{C}$ be another facet containing x_0 in its closure such that $F \subseteq \overline{F}'$. It means that $\Phi_{F'} \subseteq \Phi_F$ and $\Phi^+_{F'} \subseteq \Phi^+_F$. Let ${}_F\Theta^+_{F'}$ be the multiplicative map constructed as in 2.1 with respect to the root data attached to M_F:

$$ {}_F\Theta^+_{F'} : \mathbb{Z}[q^{\pm 1/2}][\tilde{\mathrm{X}}_*(\mathrm{T})] \longrightarrow \tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]. $$

The corresponding integral map is denoted by ${}_F\mathcal{B}^+_{F'}$ and defined by ${}_F\mathcal{B}^+_{F'}(\lambda) = q^{\ell_F(e^{\lambda})/2}\,{}_F\Theta^+_{F'}(\lambda)$ for all $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$. It satisfies ${}_F\mathcal{B}^+_{F'}(\lambda) = \tau^F_{e^{\lambda}}$ if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in (\Phi^+_F - \Phi^+_{F'}) \cup \Phi^-_{F'}$.

Remark 3.7. If $F = x_0$ then ${}_{x_0}\mathcal{B}^+_{F'} = \mathcal{B}^+_{F'}$.

Lemma 3.8. *Let $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$ be an F-positive coweight. Then ${}_F\mathcal{B}^+_{F'}(\lambda)$ lies in $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^+$ and*

$$ j^+_F({}_F\mathcal{B}^+_{F'}(\lambda)) = \mathcal{B}^+_{F'}(\lambda). \tag{3.2} $$

Proof. Decompose $\lambda = \mu - \nu$ with $\mu, \nu \in \mathscr{C}^+(F')$. Then in $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$ we have ${}_F\mathcal{B}^+_{F'}(\lambda) = q^{(\ell_F(e^{\lambda}) + \ell_F(e^{\nu}) - \ell_F(e^{\mu}))/2} \tau^F_{e^{\mu}} (\tau^F_{e^{\nu}})^{-1}$. By Lemma 2.2 applied to the pro-p Iwahori-Hecke algebra of $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$, it decomposes in $\tilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ into a linear combination of $\tau^F_{\tilde{w}}$ for $\tilde{w} \in \tilde{\mathrm{W}}_F$ where the projection w of $\tilde{w}$ in W_F satisfies $w \underset{F}{\leq} e^{\lambda}$. Fact ii ensures that those w (and $\tilde{w}$) are F-positive. Now, j_F respects the product and

$$ j^+_F({}_F\mathcal{B}^+_{F'}(\lambda)) = j_F({}_F\mathcal{B}^+_{F'}(\lambda)) = q^{(\ell_F(e^{\lambda}) + \ell_F(e^{\nu}) - \ell_F(e^{\mu}))/2} \tau_{e^{\mu}} (\tau_{e^{\nu}})^{-1} $$

because μ and ν are in particular F-positive. Apply (3.1) to conclude.

□

3.3.3. We prove Lemma 3.4. Let $\mathcal{O}$ be a $\mathfrak{W}$-orbit in $\tilde{\mathrm{X}}_*(\mathrm{T})$. Since $\mathcal{B}^+_{x_0} = \mathcal{B}^-_C$ and using (2.1), it is enough to prove

$$ \sum_{\lambda \in \mathcal{O}} \mathcal{B}^+_F(\lambda) = \sum_{\lambda \in \mathcal{O}} \mathcal{B}^+_C(\lambda) \tag{3.3} $$

for any standard facet F. If $F = x_0$ then the result is given by Proposition 3.2. Let F be a standard facet such that $F \neq x_0$.

1/ Let $\mu \in \tilde{\mathrm{X}}_*(\mathrm{T})$ be a F-positive coweight with $\mathfrak{W}_F$-orbit $\mathcal{O}_F$. We have the following identity

$$ \sum_{\mu' \in \mathcal{O}_F} \mathcal{B}^+_F(\mu') = \sum_{\mu' \in \mathcal{O}_F} j^+_F({}_F\mathcal{B}^+_F(\mu')) = \sum_{\mu' \in \mathcal{O}_F} j^+_F({}_F\mathcal{B}^+_C(\mu')) = \sum_{\mu' \in \mathcal{O}_F} \mathcal{B}^+_C(\mu') $$

where the first and third equalities come from (3.2) and the second one from Proposition 3.2 applied to M_F. .

2/ Choose ν a strongly F-positive coweight such that $\lambda+\nu$ is F-positive for all $\lambda \in \mathcal{O}$. Decompose the $\mathfrak{W}$-orbit $\mathcal{O}$ into the disjoint union of $\mathfrak{W}_F$-orbits $\mathcal{O}_F^i$ for $i \in \{1,..,r\}$. Since ν lies in both $\tilde{\mathrm{X}}_*^+(\mathrm{T})$ and $\mathscr{C}^+(F)$, we have $\mathcal{B}_F^+(-\nu) = \mathcal{B}_C^+(-\nu) = \iota_C(\tau_{e^{-\nu}})$.

Let $i \in \{1,...,r\}$ and $\lambda \in \mathcal{O}_F^i$. We have in $\tilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$

$$\mathcal{B}_F^+(\lambda) = q^{\frac{\ell(e^\lambda)-\ell(e^{\lambda+\nu})-\ell(e^\nu)}{2}} \mathcal{B}_F^+(\lambda+\nu)\mathcal{B}_F^+(-\nu).$$

Note that $\ell(e^\lambda)-\ell(e^{\lambda+\nu})-\ell(e^\nu)$ does not depend on $\lambda \in \mathcal{O}_F^i$: since $\langle \nu, \alpha\rangle = 0$ for all $\alpha \in \Phi_F^+$, this quantity is equal to $\sum_{\alpha\in\Phi^+-\Phi_F^+} |\langle\lambda,\alpha\rangle| - |\langle\lambda+\nu,\alpha\rangle| - |\langle\nu,\alpha\rangle|$ which does not depend on the choice of $\lambda \in \mathcal{O}_F^i$ because $\Phi^+ - \Phi_F^+$ is invariant under the action of $\mathfrak{W}_F$. Therefore, if we pick a representative $\lambda_i \in \lambda \in \mathcal{O}_F^i$, we have

$$\begin{aligned}
\sum_{\lambda\in\mathcal{O}_F^i} \mathcal{B}_F^+(\lambda) &= q^{\frac{\ell(e^{\lambda_i})-\ell(e^{\lambda_i+\nu})-\ell(e^\nu)}{2}} \sum_{\lambda\in\mathcal{O}_F^i} \mathcal{B}_F^+(\lambda+\nu)\mathcal{B}_C^+(-\nu).\\
&= q^{\frac{\ell(e^{\lambda_i})-\ell(e^{\lambda_i+\nu})-\ell(e^\nu)}{2}} \sum_{\lambda\in\mathcal{O}_F^i} \mathcal{B}_C^+(\lambda+\nu)\mathcal{B}_C^+(-\nu) \qquad \text{by 1/ applied to the } \mathfrak{W}_F\text{-orbit of } \lambda+\nu\\
&= \sum_{\lambda\in\mathcal{O}_F^i} \mathcal{B}_C^+(\lambda)
\end{aligned}$$

which proves that $\sum_{\lambda\in\mathcal{O}} \mathcal{B}_F^+(\lambda) = \sum_{\lambda\in\mathcal{O}} \mathcal{B}_C^+(\lambda)$.

4. Compatibility between Satake and Bernstein-type isomorphisms in characteristic p.

In this section all the algebras have coefficients in k. Let (ρ, V) be a weight and v a chosen nonzero $\tilde{\mathrm{I}}$-fixed vector. Let $\chi : \tilde{\mathfrak{H}}_k \to k$ be the associated character and F_χ the corresponding standard facet (Remark 1.9). Let $\mathbf{1}_{\mathrm{K},v} \in \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho$ be the ($\tilde{\mathrm{I}}$-invariant) function with support K and value v at 1. By [24, Lemma 3.6], the map

$$\begin{array}{ccc}
\chi \otimes_{\tilde{\mathfrak{H}}_k} \tilde{\mathrm{H}}_k & \cong & (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}} \\
1\otimes 1 & \mapsto & \mathbf{1}_{\mathrm{K},v}
\end{array} \tag{4.1}$$

induces an $\tilde{\mathrm{H}}_k$-equivariant isomorphism. Therefore,

$$\begin{array}{ccc}
\mathcal{Z}(\tilde{\mathrm{H}}_k) & \longrightarrow & \mathrm{Hom}_{\tilde{\mathrm{H}}_k}((\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}) \\
z & \longmapsto & (\mathbf{1}_{\mathrm{K},v} \mapsto \mathbf{1}_{\mathrm{K},v} z)
\end{array} \tag{4.2}$$

is a well-defined morphism of k-algebras. By [24, Corollary 3.14], passing to $\tilde{\mathrm{I}}$-invariants yields an isomorphism of k-algebras

$$\mathcal{H}(\mathrm{G},\rho) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{\mathrm{H}}_k}((\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho)^{\tilde{\mathrm{I}}}). \tag{4.3}$$

Composing (4.2) with the inverse of (4.3) therefore gives a morphism of k-algebras $\mathcal{Z}(\tilde{\mathrm{H}}_k) \to \mathcal{H}(\mathrm{G},\rho)$ and we consider its restriction to $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$:

$$\begin{array}{ccc} \mathcal{Z}^\circ(\tilde{\mathrm{H}}_k) & \longrightarrow & \mathcal{H}(\mathrm{G},\rho) \\ z & \longmapsto & (\mathbf{1}_{\mathrm{K},v} \mapsto \mathbf{1}_{\mathrm{K},v} z). \end{array} \tag{4.4}$$

For $\lambda \in \mathrm{X}_*^+(\mathrm{T})$, we denote by $\mathcal{T}'_\lambda \in \mathcal{H}(\mathrm{G},\rho)$ the image by (4.4) of the central Bernstein function z_λ defined by (2.3).

On the other hand, recall that we have the isomorphism ([24, Theorem 4.11])

$$\mathcal{T}: \quad k[\mathrm{X}_*^+(\mathrm{T})] \quad \xrightarrow{\simeq} \quad \mathcal{H}(\mathrm{G},\rho) \tag{1.2}$$

where $\mathcal{T}_\lambda$ for $\lambda \in \mathrm{X}_*^+(\mathrm{T})$ is defined by

$$\mathcal{T}_\lambda : \mathbf{1}_{\mathrm{K},v} \mapsto \mathbf{1}_{\mathrm{K},v}\mathcal{B}^+_{F_\chi}(\lambda). \tag{4.5}$$

Proposition 4.1. *We have $\mathcal{T}'_\lambda = \mathcal{T}_\lambda$ for all $\lambda \in \mathrm{X}_*^+(\mathrm{T})$.*

Proof. It is enough to check that these operators coincide on $\mathbf{1}_{\mathrm{K},v}$. Recall that $\mathcal{O}(\lambda)$ denotes the $\mathfrak{W}$-orbit of λ.

a/ Let $\lambda' \in \mathcal{O}(\lambda)$ and suppose that $\lambda' \neq \lambda$. By (2.4), we have $\mathcal{B}^+_{F_\chi}(\lambda')\mathcal{B}^+_{F_\chi}(\lambda) = \mathcal{B}^+_{F_\chi}(\lambda)\mathcal{B}^+_{F_\chi}(\lambda') = 0$ in $\tilde{\mathrm{H}}_k$. It implies that $\mathcal{T}_\lambda(\mathbf{1}_{\mathrm{K},v}\mathcal{B}^+_{F_\chi}(\lambda')) = 0$ and therefore that $\mathbf{1}_{\mathrm{K},v}\mathcal{B}^+_{F_\chi}(\lambda') = 0$ by [17, Corollary 6.5] that claims that $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho$ is a torsion free $\mathcal{H}(\mathrm{G},\rho)$-module.

b/ By Lemma 3.4, we have

$$\begin{aligned} \mathcal{T}'_\lambda(\mathbf{1}_{\mathrm{K},v}) &= \mathbf{1}_{\mathrm{K},v}\mathcal{B}^+_{F_\chi}(\lambda') + \sum_{\lambda' \in \mathcal{O}(\lambda), \lambda' \neq \lambda} \mathbf{1}_{\mathrm{K},v}\mathcal{B}^+_{F_\chi}(\lambda') \\ &= \mathcal{T}_\lambda(\mathbf{1}_{\mathrm{K},v}) + \sum_{\lambda' \in \mathcal{O}(\lambda), \lambda' \neq \lambda} \mathbf{1}_{\mathrm{K},v}\mathcal{B}^+_{F_\chi}(\lambda') \\ &= \mathcal{T}_\lambda(\mathbf{1}_{\mathrm{K},v}) \text{ by a/}. \end{aligned}$$

□

Proposition 4.1 implies:

Theorem 4.2. *The diagram*

$$\begin{tikzcd} k[\mathrm{X}_*^+(\mathrm{T})] \arrow[r, "(2.7)"] \arrow[d, equal] & \mathcal{Z}^\circ(\tilde{\mathrm{H}}_k) \arrow[d, "(4.4)"] \\ k[\mathrm{X}_*^+(\mathrm{T})] \arrow[r, "\mathcal{T}"] & \mathcal{H}(\mathrm{G},\rho) \end{tikzcd} \tag{4.6}$$

is a commutative diagram of isomorphisms of k-algebras.

Remark that we have not used the fact that (2.7) is multiplicative. We proved this fact beforehand in Proposition 2.8. It can also be seen as a consequence of the commutativity of the diagram.

5. Supersingularity

We turn to the study of the $\tilde{\mathrm{H}}_k$-modules with finite length. We consider left modules unless otherwise specified. Recall that k is algebraically closed.

5.1. **A basis for the generic pro-p Iwahori Hecke ring.** We recall the $\mathbb{Z}$-basis for $\tilde{\mathrm{H}}_\mathbb{Z}$ defined in [31]. It is indexed by $w \in \tilde{\mathrm{W}}$ and is denoted by $(E_w^+)_{w\in\tilde{\mathrm{W}}}$ in [31]. We will call it $(\mathcal{B}_C^+(w))_{w\in\tilde{\mathrm{W}}}$ because it coincides on $\tilde{\mathrm{X}}_*(\mathrm{T})$ with the definition introduced in 2.1. Recall that we have a decomposition of $\tilde{\mathrm{W}}$ as the semidirect product:

$$\tilde{\mathrm{W}} = \tilde{\mathfrak{W}} \ltimes \mathrm{X}_*(\mathrm{T}).$$

For $w_0 \in \tilde{\mathfrak{W}}$ set $\mathcal{B}_C^+(w_0) = \tau_{w_0}$ and for $w = w_0 x \in \tilde{\mathfrak{W}} \ltimes \mathrm{X}_*(\mathrm{T})$, define in $\tilde{\mathrm{H}}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}[q^{\pm 1/2}]$:

$$\mathcal{B}_C^+(w) = q^{(\ell(w)-\ell(w_0)-\ell(x))/2}\mathcal{B}_C^+(w_0)\mathcal{B}_C^+(x) = q^{(\ell(w)-\ell(w_0))/2}\tau_{w_0}\Theta_C^+(x).$$

By [31, Thm 2 and Prop 8], this element lies in $\tilde{\mathrm{H}}_\mathbb{Z}$ and the set of all $(\mathcal{B}_C^+(w))_{w\in\tilde{\mathrm{W}}}$ is a $\mathbb{Z}$-basis for $\tilde{\mathrm{H}}_\mathbb{Z}$.

Remark 5.1. Let $d \in \mathcal{D}$ and $\tilde{d} \in \tilde{\mathrm{W}}$ a lift for d. Write (Proposition 1.3) $\tilde{d} = w_0 e^{-\lambda}$ with $w_0 \in \tilde{\mathfrak{W}}$ and $\lambda \in \mathrm{X}_*^+(\mathrm{T})$ with $\ell(e^\lambda) = \ell(d^{-1}) + \ell(w_0)$. Then in $\tilde{\mathrm{H}}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}[q^{\pm 1/2}]$, we have

$$\mathcal{B}_C^+(\tilde{d}) = q^{(\ell(d)-\ell(w_0)+\ell(e^\lambda))/2}\tau_{w_0}\tau_{e^\lambda}^{-1} = q^{\ell(d)}\tau_{\tilde{d}^{-1}}^{-1} = (-1)^{\ell(d)}\iota(\tau_{\tilde{d}}). \tag{5.1}$$

5.2. **Topology on the pro-p Iwahori Hecke algebra in characteristic p.** We consider the (finitely generated) ideal $\mathfrak{I}$ of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ generated by all z_λ for $\lambda \in \mathrm{X}_*^+(\mathrm{T})$ such that $\ell(e^\lambda) > 0$ and the associated ring filtration of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$. Any $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$-module M can be endowed with the $\mathfrak{I}$-adic topology induced by the filtration

$$M \supseteq \mathfrak{I}M \supseteq \mathfrak{I}^2 M \supseteq \ldots$$

An example of such a module is $\tilde{\mathrm{H}}_k$ itself.

We define on $\tilde{\mathrm{H}}_k$ another decreasing filtration $(F_n\tilde{\mathrm{H}}_k)_{n\in\mathbb{N}}$ by k-vector spaces where

$$F_n\tilde{\mathrm{H}}_k := k\text{-vector space generated by all } \mathcal{B}_C^+(w) \text{ for } w \in \tilde{\mathrm{W}} \text{ such that } \ell(w) \geq n. \tag{5.2}$$

Lemma 5.2. *The filtration (5.2) is a filtration of $\tilde{\mathrm{H}}_k$ as a right $\mathcal{A}_C^+$-module. In particular, it is a filtration of $\tilde{\mathrm{H}}_k$ as a (left and right) $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$-module. This filtration is compatible with the $\mathfrak{I}$-filtration: for all $n \in \mathbb{N}$, we have*

$$\mathfrak{I}\, F_n\tilde{\mathrm{H}}_k \subseteq F_{n+1}\tilde{\mathrm{H}}_k.$$

Proof. Let $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$ and $w \in \tilde{\mathrm{W}}$. In $\tilde{\mathrm{H}}_k$, we have $\mathcal{B}_C^+(w)\mathcal{B}_C^+(\lambda) = 0$ if $\ell(w) + \ell(e^\lambda) > \ell(we^\lambda)$ and $\mathcal{B}_C^+(w)\mathcal{B}_C^+(\lambda) = \mathcal{B}_C^+(we^\lambda)$ if $\ell(w) + \ell(e^\lambda) = \ell(we^\lambda)$. It proves the claims.

□

Proposition 5.3. *The $\mathfrak{I}$-adic topology on $\tilde{\mathrm{H}}_k$ is equivalent to the topology on $\tilde{\mathrm{H}}_k$ induced by the filtration $(F_n\tilde{\mathrm{H}}_k)_{n\in\mathbb{N}}$. In particular, it is independent of the choice of the uniformizer ϖ.*

Proof. The previous lemma ensures that the $\mathfrak{I}$-adic topology on $\tilde{\mathrm{H}}_k$ is stronger than the topology induced by $(F_n\tilde{\mathrm{H}}_k)_{n\in\mathbb{N}}$. We have to prove that given $m \in \mathbb{N}$, $m \geq 1$, there is $n \in \mathbb{N}$ such that $F_n\tilde{\mathrm{H}}_k \subseteq \mathfrak{I}^m\tilde{\mathrm{H}}_k$.

Fact iii. *For $\lambda \in \mathrm{X}_*(\mathrm{T})$ such that $\ell(e^\lambda) > 0$ and for $m \geq 1$, we have $\mathcal{B}_C^+((m+1)\lambda) \in \mathfrak{I}^m\tilde{\mathrm{H}}_k$.*

Proof of the fact. We check that for $m \in \mathbb{N}$ we have $\mathcal{B}_C^+((m+1)\lambda) = z_\lambda^m\mathcal{B}_C^+(\lambda)$. Notice that $\mathcal{B}_C^+(2\lambda) = \mathcal{B}_C^+(\lambda)\mathcal{B}_C^+(\lambda) = z_\lambda\mathcal{B}_C^+(\lambda)$ by (2.4). Now let $m \geq 2$. We have $\mathcal{B}_C^+((m+1)\lambda) = \mathcal{B}_C^+(m\lambda)\mathcal{B}_C^+(\lambda) = z_\lambda^m\mathcal{B}_C^+(\lambda)$ by induction.

□

Fact iv. *Let $m \geq 1$. There is $A_m \in \mathbb{N}$ such that for any $\lambda \in \mathrm{X}_*(\mathrm{T})$, if $\ell(e^\lambda) > A_m$ then $\mathcal{B}_C^+(\lambda) \in \mathfrak{I}^m\tilde{\mathrm{H}}_k$.*

Proof of the fact. Let $\{z_{\lambda_1}, \ldots, z_{\lambda_r}\}$ be a system of generators of $\mathfrak{I}$. Set $A_m := m\sum_{i=1}^r \ell(e^{\lambda_i})$. Let $\lambda \in \mathrm{X}_*(\mathrm{T})$ such that $\ell(e^\lambda) > 0$. It is $\mathfrak{W}$-conjugate to an element $\lambda_0 \in \mathrm{X}_*^+(\mathrm{T})$ and one can write $\lambda = w_0\lambda_0w_0^{-1}$ with $w_0 \in \mathfrak{W}$ and $\lambda_0 = \sum_{i=1}^r a_i\lambda_i$ with $a_i \in \mathbb{N}$ (not all equal to zero). If $\ell(e^\lambda) = \ell(e^{\lambda_0}) > A_m$, then there is $i_0 \in \{1, ..., r\}$ such that $a_{i_0} > m$ and $\mathcal{B}_C^+(\lambda) = \prod_{i=1}^r \mathcal{B}_C^+(a_i(w_0.\lambda_i)) \in \mathcal{B}_C^+((m+1)(w_0.\lambda_{i_0}))\tilde{\mathrm{H}}_k \subseteq \mathfrak{I}^m\tilde{\mathrm{H}}_k$ by Fact iii. □

We know turn to the proof of the proposition. To any $w_0 \in \mathfrak{W}$ corresponds, by [30, (1.6.3)], a finite set $\mathrm{X}(w_0)$ of elements in $\mathrm{X}_*(\mathrm{T})$ such that

$$\text{for all } \lambda \in \mathrm{X}_*(\mathrm{T}) \text{ there is } \mu \in \mathrm{X}(w_0) \text{ such that } \ell(w_0e^\lambda) = \ell(w_0e^\mu) + \ell(e^{\lambda-\mu}).$$

Let $\tilde{w} \in \tilde{\mathrm{W}}$ with image w_0 by the projection $\tilde{\mathrm{W}} \to \mathfrak{W}$. Its projection w in W has the form $w = w_0e^\lambda \in \mathfrak{W} \ltimes \mathrm{X}_*(\mathrm{T})$ and there is $\mu \in \mathrm{X}(w_0)$ such that $\ell(w) = \ell(w_0e^\mu) + \ell(e^{\lambda-\mu})$. Choose lifts $\widetilde{w_0e^\mu}$ and $\widetilde{e^{\lambda-\mu}}$ in $\tilde{\mathrm{W}}$ for w_0e^μ and $e^{\lambda-\mu}$. The product $\widetilde{w_0e^\mu}\widetilde{e^{\lambda-\mu}}$ differs from $\tilde{w}$ by an element

in $\mathrm{T}^0/\mathrm{T}^1$ (which has length zero). Therefore, $\mathcal{B}_C^+(\tilde{w}) \in \tilde{\mathrm{H}}_k\mathcal{B}_C^+(\lambda-\mu)$ (see the proof of Lemma 5.2 for example). If $\ell(\tilde{w}) > A_m(w_0) := A_m + \max\{\ell(w_0e^\mu),\ \mu\in \mathrm{X}(w_0)\}$ then $\ell(e^{\lambda-\mu}) > A_m$ and $\mathcal{B}_C^+(\tilde{w}) \in \mathfrak{I}^m\tilde{\mathrm{H}}_k$ by Fact iv. We have proved that $n \geq \max\{A_m(w_0), w_0 \in \mathfrak{W})\}$ implies $F_n\tilde{\mathrm{H}}_k \subseteq \mathfrak{I}^m\tilde{\mathrm{H}}_k$.

□

5.3. **The category of finite length modules over the pro-p Iwahori Hecke algebra in characteristic p.** We consider the abelian category $\mathrm{Mod}_{fg}(\tilde{\mathrm{H}}_k)$ of all $\tilde{\mathrm{H}}_k$-modules with finite length.

For a $\tilde{\mathrm{H}}_k$-module, having finite length is equivalent to being finite dimensional as a k-vector space ([32, 5.3] or [23, Lemma 6.14]). Therefore, any irreducible $\tilde{\mathrm{H}}_k$-module is finite dimensional and has a central character, and any module in $\mathrm{Mod}_{fg}(\tilde{\mathrm{H}}_k)$ decomposes uniquely into a direct sum of indecomposable modules.

5.3.1. *The category of finite dimensional $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$-modules.* Let $\mathrm{Mod}_{fd}(\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k))$ denote the category of finite dimensional $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$-modules. For $\mathfrak{M}$ a maximal ideal of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$, we consider the full subcategory

$$\mathfrak{M}-\mathrm{Mod}_{fd}(\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k))$$

of the modules M of $\mathfrak{M}$-torsion, that is to say such that there is $e\in\mathbb{N}$ satisfying $\mathfrak{M}^eM=0$. The category $\mathrm{Mod}_{fd}(\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k))$ decomposes into the direct sum of all $\mathfrak{M}-\mathrm{Mod}_{fd}(\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k))$ where $\mathfrak{M}$-ranges over the maximal ideals of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$.

5.3.2. *Blocks of $\tilde{\mathrm{H}}_k$-modules with finite length.* For $\mathfrak{M}$ a maximal ideal of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$, we say that a $\tilde{\mathrm{H}}_k$-module with finite length is a $\mathfrak{M}$-torsion module if its restriction to a $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$-module lies in the subcategory $\mathfrak{M}-\mathrm{Mod}_{fd}(\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k))$. We denote by

$$\mathfrak{M}-\mathrm{Mod}_{fg}(\tilde{\mathrm{H}}_k) \tag{5.3}$$

the full subcategory of $\mathrm{Mod}_{fg}(\tilde{\mathrm{H}}_k)$ of such modules.

Lemma 5.4. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two maximal ideals of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$. If there is a nonzero $\mathfrak{M}$-torsion module M and a nonzero $\mathfrak{N}$-torsion module N such that $\mathrm{Ext}^r_{\tilde{\mathrm{H}}_k}(M,N)\neq 0$ for some $r\geq 0$, then $\mathfrak{M}=\mathfrak{N}$.*

Proof. For any $\tilde{\mathrm{H}}_k$-modules X and Y, the natural morphisms of algebras $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)\to\mathrm{End}_{\tilde{\mathrm{H}}_k}(X)$ and $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)\to\mathrm{End}_{\tilde{\mathrm{H}}_k}(Y)$ equip $\mathrm{Hom}_{\tilde{\mathrm{H}}_k}(X,Y)$ with a structure of central $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$-bimodule. The space $\mathrm{Ext}^r(M,N)$ is therefore naturally a central $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$-bimodule. It is an $\mathfrak{M}$-torsion module and a $\mathfrak{N}$-torsion module. Therefore it is zero unless $\mathfrak{M}=\mathfrak{N}$.

□

For $\mathfrak{M}$ a maximal ideal of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$, we call $\mathfrak{M}-\mathrm{Mod}_{fd}(\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k))$ the block corresponding to $\mathfrak{M}$. Since $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ is a central finitely generated subalgebra of $\tilde{\mathrm{H}}_k$, an indecomposable $\tilde{\mathrm{H}}_k$-module with finite length is a $\mathfrak{M}$-torsion module for some maximal ideal $\mathfrak{M}$ of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$.

Remark 5.5. A $\tilde{\mathrm{H}}_k$-module with finite length M lies in the block corresponding to some maximal ideal $\mathfrak{M}$ if and only if all the characters of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ contained in M have kernel $\mathfrak{M}$.

Remark 5.6. The blocks (5.3) are not indecomposable. They can for example be further decomposed *via* the idempotents introduced in 1.2.8.

5.3.3. *The supersingular block.*

Definition 5.7. We call a maximal ideal $\mathfrak{M}$ of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ supersingular if it contains the ideal $\mathfrak{I}$ defined in 5.2. A character of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ is called supersingular if its kernel is a supersingular maximal ideal of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$.

Given a character ω of the connected center Z of G, there is a unique supersingular character ζ_ω of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ satisfying $\zeta_\omega(z_\lambda)=\omega(\lambda(\varpi))$ for any $\lambda\in\mathrm{X}_*^+(\mathrm{T})$ with length zero. A character of the center of $\tilde{\mathrm{H}}_k$ is called "null" in [31] if it takes value zero at all central elements (2.2) for all $\mathfrak{W}$-orbits $\mathcal{O}$ in $\tilde{\mathrm{X}}_*(\mathrm{T})$ containing a coweight with length $\neq 0$.

Lemma 5.8. *A character $\mathcal{Z}(\tilde{\mathrm{H}}_k)\to k$ is "null" if and only if its restriction to $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ is a supersingular character in the sense of Definition 5.7.*

Proof. Consider a character $\zeta:\mathcal{Z}(\tilde{\mathrm{H}}_k)\to k$ whose restriction to $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ is supersingular. We want to prove that ζ is "null". The $\tilde{\mathrm{H}}_k$-module $\tilde{\mathrm{H}}_k\otimes_{\mathcal{Z}(\tilde{\mathrm{H}}_k)}\zeta$ being finite dimensional, it contains a character $\hat{\zeta}$ for the commutative finitely generated k-algebra $(\mathcal{A}_C^+)_k$ and the restriction of $\hat{\zeta}$ to $\mathcal{Z}(\tilde{\mathrm{H}}_k)$ coincides with ζ. We look at the restriction of $\hat{\zeta}$ to $(\mathcal{A}_C^+)_k$. Let $\lambda\in\mathrm{X}_*^+(\mathrm{T})$ with $\ell(e^\lambda)\neq 0$; by (2.4), there is at most one $\mathfrak{W}$-conjugate λ' of λ such that $\hat{\zeta}(\mathcal{B}_C^+(\lambda'))\neq 0$ and if there exists such a λ', then $\hat{\zeta}(z_\lambda)=\zeta(z_\lambda)\neq 0$, which is a contradiction: we have proved that $\hat{\zeta}(\mathcal{B}_C^+(\lambda'))=0$ for all $\lambda'\in\mathrm{X}_*(\mathrm{T})$ with $\ell(e^{\lambda'})\neq 0$ which implies that it is also the case for $\lambda'\in\tilde{\mathrm{X}}_*(\mathrm{T})$ with $\ell(e^{\lambda'})\neq 0$. Therefore, ζ is "null".

□

A finite dimensional $\tilde{\mathrm{H}}_k$-module M with central character is called supersingular in [31] if this central character is"null". We extend this definition.

Proposition-Definition 5.9. *An indecomposable finite length $\tilde{\mathrm{H}}_k$-module is in the supersingular block and is called* supersingular *if and only if equipped with the discrete topology, it is a continuous module for the $\mathfrak{I}$-adic topology on $\tilde{\mathrm{H}}_k$ or equivalently, for the topology induced by the filtration* (5.2)*.*
The notion of supersingularity is independent of the choice of the uniformizer ϖ.

Proof. An indecomposable $\tilde{\mathrm{H}}_k$-module M with finite length is in the supersingular block if and only if there is $m \geq 1$ such that $\mathfrak{J}^m M = \{0\}$. Then use Proposition 5.3. $\square$

Remark 5.10. If $\mathbf{G}$ is of adjoint type or $\mathbf{G} = \mathrm{GL}_n$, then by Corollary 2.7 and the proof of Proposition 2.6, the ideal $\mathfrak{J}$ and therefore the notion of supersingularity for the finite length $\tilde{\mathrm{H}}_k$-modules is independent of all the choices.

Remark 5.11. Suppose that $\mathbf{G} = \mathrm{GL}_n$. Then we get from [31, Theorem 5] and [22, Theorem 7.3] that an indecomposable $\tilde{\mathrm{H}}_k$-module with finite length is in the supersingular block if and only if it contains a character for the affine Hecke subalgebra $\tilde{\mathrm{H}}_k^{aff}$. So we can also deduce from this that the notion of supersingularity for a $\tilde{\mathrm{H}}_k$-module with finite length is independent of all the choices.

We generalize [22, Theorem 7.3]:

Theorem 5.12. *A finite length $\tilde{\mathrm{H}}_k$-module in the supersingular block contains a character for the affine Hecke subalgebra $\tilde{\mathrm{H}}_k^{aff}$.*

Proof. Let M be an $\tilde{\mathrm{H}}_k$-module with finite length in the supersingular block. By the previous proposition, there is $n \in \mathbb{N}$ such that for any $w \in \tilde{\mathrm{W}}$, if $\ell(w) > n$ then $\mathcal{B}_C^+(w)M = 0$. Let $x \in M$ supporting a character for $\tilde{\mathfrak{H}}_k$ (see 1.2.9) and let $d \in \mathcal{D}$ with maximal length such that $\mathcal{B}_C^+(\tilde{d})x \neq 0$ where $\tilde{d} \in \tilde{\mathrm{W}}$ denotes a lift for d (the property $\mathcal{B}_C^+(\tilde{d})x \neq 0$ does not depend on the choice of the lift $\tilde{d}$). As in [22, Theorem 7.3], we prove that $x' := \mathcal{B}_C^+(\tilde{d})x$ supports a character for $\tilde{\mathrm{H}}_k^{aff}$ which is generated by all τ_t and all $\tau_{\tilde{s}}$ for $t \in \mathrm{T}^0/\mathrm{T}^1$ and $s \in S_{aff}$ with chosen lift $\tilde{s} \in \tilde{\mathrm{W}}$. From the relations (1.10) we get that $\tau_t x' = \mathcal{B}_C^+(\tilde{d})\tau_{d^{-1}td}x$ is proportional to x'. Now let $s \in S_{aff}$. If $\ell(sd) = \ell(d) - 1$, then $sd \in \mathcal{D}$ after Proposition 1.3 and, by (5.1), the element x' is equal to $\iota(\tau_{\tilde{s}})\iota(\tau_{\tilde{s}\tilde{d}})x$ (up to an invertible element in k) so that $\tau_{\tilde{s}}x' = 0$ by Remark 1.8. If $\ell(sd) = \ell(d) + 1$ and $sd \in \mathcal{D}$, then $\mathcal{B}_C^+(\tilde{s}\tilde{d})x$ is equal to zero on one side, and to $\iota(\tau_{\tilde{s}})x'$ (up to an invertible element in k) by (5.1) on the other side. It proves that $\tau_{\tilde{s}}x'$ is proportional to x' by Remark 1.8. If $\ell(sd) = \ell(d)+1$ and $sd \notin \mathcal{D}$ then there is $s' \in S$ such that $sd = ds'$ by Proposition 1.3, and $\iota(\tau_{\tilde{s}})x'$ is proportional to $\mathcal{B}_C^+(\tilde{d})\iota(\tau_{\tilde{s}'})x$ and therefore to x' because $\iota(\tau_{\tilde{s}'}) \in \tilde{\mathfrak{H}}_k$. $\square$

5.4. Pro-p-Iwahori invariants of parabolic inductions and special representations of G in characteristic p.

5.4.1. In this paragraph, $\mathbf{k}$ is an arbitrary field. Let F be a standard facet, Π_F the associated set of simple roots and P_F the corresponding standard parabolic subgroup with Levi decomposition $\mathrm{P}_F = \mathrm{M}_F\mathrm{N}_F$. Recall that by Remark 3.6 the subspace $\tilde{\mathrm{H}}_{\mathbf{k}}(\mathrm{M}_F)^-$ of $\tilde{\mathrm{H}}_{\mathbf{k}}(\mathrm{M}_F)$ generated over $\mathbf{k}$ by all τ_w^F for all F-negative $w \in \tilde{\mathrm{W}}_F$ injects in $\tilde{\mathrm{H}}_{\mathbf{k}}$ *via*

$$\begin{array}{cccc} j_F^- : & \tilde{\mathrm{H}}_{\mathbf{k}}(\mathrm{M}_F)^- & \longrightarrow & \tilde{\mathrm{H}}_{\mathbf{k}} \\ & \tau_w^F & \longmapsto & \tau_w \end{array}.$$

It endows $\tilde{\mathrm{H}}_k$ with a structure of left module over $\tilde{\mathrm{H}}_{\mathbf{k}}(\mathrm{M}_F)^-$.

Proposition 5.13. *Let $(\sigma, \mathrm{V}_\sigma)$ be a smooth $\mathbf{k}$-representation of M_F. We consider the parabolic induction $\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}}\sigma$ and its $\tilde{\mathrm{I}}$-invariant subspace $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}}\sigma)^{\tilde{\mathrm{I}}}$. We have a surjective morphism of right $\tilde{\mathrm{H}}_{\mathbf{k}}$-modules*

$$\sigma^{\tilde{\mathrm{I}}_F} \otimes_{\tilde{\mathrm{H}}_{\mathbf{k}}(\mathrm{M}_F)^-} \tilde{\mathrm{H}}_{\mathbf{k}} \longrightarrow (\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}}\sigma)^{\tilde{\mathrm{I}}} \tag{5.4}$$

sending $v \otimes 1$ to the unique $\tilde{\mathrm{I}}$-invariant function with support in $\mathrm{P}_F\tilde{\mathrm{I}}$ and value v at 1_{G}.

Remark 5.14. In the case of $\mathbf{G} = \mathrm{PGL}_n$ or GL_n Proposition 5.2 in [22] implies that (5.4) is an isomorphism. The proof uses the Bernstein integral relations for the minuscule coweights. We think that the bijectivity is true for a general (split) $\mathbf{G}$, but we will only use the surjectivity here.

The proposition follows from the discussion below. All the lemmas are proved in the next paragraph.

The unipotent subgroup N_F is generated by all the root subgroups $\mathcal{U}_\alpha$ for $\alpha \in \Phi^+ - \Phi_F^+$. Let N_F^- denote the opposite unipotent subgroup of G. The pro-p Iwahori subgroup $\tilde{\mathrm{I}}$ has the following decomposition:

$$\tilde{\mathrm{I}} = \tilde{\mathrm{I}}_F^+\, \tilde{\mathrm{I}}_F^0\, \tilde{\mathrm{I}}_F^- \text{ where } \tilde{\mathrm{I}}_F^+ := \tilde{\mathrm{I}} \cap \mathrm{N}_F,\ \tilde{\mathrm{I}}_F := \tilde{\mathrm{I}} \cap \mathrm{M}_F,\ \tilde{\mathrm{I}}_F^- := \tilde{\mathrm{I}} \cap \mathrm{N}_F^-.$$

Lemma 5.15. *Let $\mathcal{D}_F = \{d \in \mathfrak{W},\ d^{-1}\Phi_F^+ \subseteq \Phi^+\}$.*

i. *For $d \in \mathcal{D}_F$, we have $\mathrm{P}_F\tilde{\mathrm{I}}\hat{d}\,\tilde{\mathrm{I}} = \mathrm{P}_F\hat{d}\,\tilde{\mathrm{I}}$.*
ii. *The set of all lifts $\hat{w} \in \mathrm{G}$ for $w \in \mathcal{D}_F$ is a system of representatives of the double cosets $\mathrm{P}_F\backslash \mathrm{G}/\tilde{\mathrm{I}}$.*
iii. *For $d \in \mathcal{D}_F$, let $\tilde{\mathrm{I}}\hat{d}\,\tilde{\mathrm{I}} = \coprod_y \tilde{\mathrm{I}}\hat{d}y$ be a decomposition in right cosets. Then*
$$\mathrm{P}_F\hat{d}\,\tilde{\mathrm{I}} = \coprod_y \mathrm{P}_F\tilde{\mathrm{I}}\hat{d}y.$$
iv. *Let $d \in \mathcal{D}_F$. By the projection $\mathrm{P}_F \twoheadrightarrow \mathrm{M}_F$, the image of $\mathrm{P}_F \cap \hat{d}\,\tilde{\mathrm{I}}\hat{d}^{-1}$ is $\tilde{\mathrm{I}}_F$.*

An element $m \in \mathrm{M}_F$ contracts $\tilde{\mathrm{I}}_F^+$ and dilates $\tilde{\mathrm{I}}_F^-$ if it satisfies the conditions (see [7, (6.5)]):

$$m\tilde{\mathrm{I}}_F^+m^{-1} \subseteq \tilde{\mathrm{I}}_F^+, \qquad m^{-1}\tilde{\mathrm{I}}_F^-m \subseteq \tilde{\mathrm{I}}_F^-. \tag{5.5}$$

Remark 5.16. This property of an element $m \in \mathrm{M}_F$ only depends on the double coset $(\mathrm{M}_F \cap \tilde{\mathrm{I}})m(\mathrm{M}_F \cap \tilde{\mathrm{I}})$. Furthermore, if $m \in \mathrm{K} \cap \mathrm{M}_F$ then $m\tilde{\mathrm{I}}_F^+m^{-1} = \tilde{\mathrm{I}}_F^+$ and $m^{-1}\tilde{\mathrm{I}}_F^-m = \tilde{\mathrm{I}}_F^-$.

Lemma 5.17. *Let $w \in \tilde{\mathrm{W}}_F$. The element $\hat{w}$ satisfies (5.5) if and only if w is F-negative.*

Let $(\sigma, \mathrm{V}_\sigma)$ as in the proposition. Let $v \in \mathrm{V}_\sigma^{\tilde{\mathrm{I}}_F}$ and $d \in \mathcal{D}_F$. By Lemma 5.15 ii and iv, the $\tilde{\mathrm{I}}$-invariant function

$$f_{d,v} \in (\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} \sigma)^{\tilde{\mathrm{I}}}$$

with support in $\mathrm{P}_F \hat{d} \tilde{\mathrm{I}}$ and value v at $\hat{d}$ is well defined and the set of all $f_{d,v}$ form a basis of $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} \sigma)^{\tilde{\mathrm{I}}}$, when d ranges over $\mathcal{D}_F$ and v over a basis of $\sigma^{\tilde{\mathrm{I}}_F}$.

Lemma 5.18. i. *Let w an F-negative element in $\tilde{\mathrm{W}}_F$. Then $f_{1,v} \,.\, \tau_w = f_{1,\, v.\tau_w^F}$.*
ii. *We have $f_{1,v} \,.\, \tau_d = f_{d,v}$.*

5.4.2. *Proof of the Lemmas.* Recall that given $\alpha \in \Phi$, the root subgroup $\mathcal{U}_\alpha$ is endowed with a filtration $\mathcal{U}_{(\alpha,k)}$ for $k \in \mathbb{Z}$ (see for example [27, I.1] or [23, 4.2]). The product map

$$\prod_{\alpha \in \Phi^-} \mathcal{U}_{(\alpha,1)} \times \mathrm{T}^1 \times \prod_{\alpha \in \Phi^+} \mathcal{U}_{(\alpha,0)} \xrightarrow{\sim} \tilde{\mathrm{I}} \tag{5.6}$$

induces a bijection, where the products on the left hand side are ordered in some arbitrary chosen way ([27, Proposition I.2.2]). The subgroup $\tilde{\mathrm{I}}_F^+$ (resp. $\tilde{\mathrm{I}}_F^-$) of $\tilde{\mathrm{I}}$ is generated by the image of $\prod_{\alpha \in \Phi^+ - \Phi_F^+} \mathcal{U}_{(\alpha,0)}$ (resp. $\prod_{\alpha \in \Phi^- - \Phi_F^-} \mathcal{U}_{(\alpha,1)}$). The subgroup $\tilde{\mathrm{I}}_F$ of $\tilde{\mathrm{I}}$ is generated by T^1 and the image of $\prod_{\alpha \in \Phi_F^-} \mathcal{U}_{(\alpha,1)} \times \prod_{\alpha \in \Phi_F^+} \mathcal{U}_{(\alpha,0)}$.

Proof of Lemma 5.15. i. We have $\mathrm{P}_F \tilde{\mathrm{I}} \hat{d} \tilde{\mathrm{I}} = \mathrm{P}_F \tilde{\mathrm{I}}_F^- \hat{d} \tilde{\mathrm{I}}$. But for $\alpha \in \Phi^+$, we have $\hat{d}^{-1} \,\mathcal{U}_{(-\alpha,1)}\, \hat{d} = \mathcal{U}_{(-d^{-1}\alpha,1)} \subseteq \tilde{\mathrm{I}}$ so $\tilde{\mathrm{I}}_F^- \hat{d} \subseteq \hat{d} \tilde{\mathrm{I}}$ so $\mathrm{P}_F \tilde{\mathrm{I}} \hat{d} \tilde{\mathrm{I}} = \mathrm{P}_F \hat{d} \tilde{\mathrm{I}}$. Point ii follows by Bruhat decomposition for K and Iwasawa decomposition for G. For iii, we first recall that the image of $\mathrm{P}_F \cap \mathrm{K}$ by the reduction $red: \mathrm{K} \to \overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$ modulo K_1 is the group $\overline{\mathbf{P}}(\mathbb{F}_q)$ of $\mathbb{F}_q$-points of a parabolic subgroup containing $\overline{\mathbf{B}}$ (notations in 1.2). Recall that the Weyl group of $\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$ is $\mathfrak{W}$: for $w \in \mathfrak{W}$ we will still denote by w with a chosen lift in $\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$. The set $\mathcal{D}_F$ is a system of representatives of $\overline{\mathbf{P}}(\mathbb{F}_q) \backslash \overline{\mathbf{G}}_{x_0}(\mathbb{F}_q) / \overline{\mathbf{U}}(\mathbb{F}_q)$. For $d \in \mathcal{D}_F$, we have using [8, 2.5.12]

$$\overline{\mathbf{P}}(\mathbb{F}_q) \cap d \overline{\mathbf{U}}(\mathbb{F}_q) d^{-1} \subset \overline{\mathbf{U}}(\mathbb{F}_q)$$

We deduce that the image of $\mathrm{P}_F \cap \tilde{\mathrm{I}}_F^- d \tilde{\mathrm{I}} d^{-1}$ by red is contained in $\overline{\mathbf{U}}(\mathbb{F}_q)$ and therefore $\mathrm{P}_F \cap \tilde{\mathrm{I}}_F^- \hat{d} \tilde{\mathrm{I}} \hat{d}^{-1} \subset \tilde{\mathrm{I}}$.
Now let $d \in \mathcal{D}_F$ and $y \in \tilde{\mathrm{I}}$. By the previous observations, $\hat{d} \in \mathrm{P}_F \tilde{\mathrm{I}} \hat{d} y = \mathrm{P}_F \tilde{\mathrm{I}}_F^- \hat{d} y$ implies $\hat{d} \in \tilde{\mathrm{I}} \hat{d} y$. It proves iii. In passing we proved that $\mathrm{P}_F \cap \hat{d} \tilde{\mathrm{I}} \hat{d}^{-1}$ is contained in $\mathrm{P}_F \cap \tilde{\mathrm{I}} = \tilde{\mathrm{I}}_F \tilde{\mathrm{I}}_F^+$ and $\tilde{\mathrm{I}}_F$ is contained in $\mathrm{P}_F \cap \hat{d} \tilde{\mathrm{I}} \hat{d}^{-1}$ by definition of $\mathcal{D}_F$. It proves iv. $\square$

Proof of Lemma 5.17. By Remark 5.16 it is enough to prove the fact for $w = e^\lambda \in \mathrm{X}_*(\mathrm{T})$. A lift for e^λ is given by $\lambda(\varpi^{-1})$. The element $\lambda(\varpi^{-1})$ satisfies (5.5) if

$$\text{for all } \alpha \in \Phi^+ - \Phi_F^+ \text{ we have } \lambda(\varpi^{-1}) \mathcal{U}_{(\alpha,0)} \lambda(\varpi) \subseteq \tilde{\mathrm{I}}_F^+ \text{ and } \lambda(\varpi) \mathcal{U}_{(-\alpha,1)} \lambda(\varpi^{-1}) \subseteq \tilde{\mathrm{I}}_F^-. \tag{5.7}$$

By [23, Remark 4.1(1)] (for example), $\lambda(\varpi^{-1}) \mathcal{U}_{(\alpha,0)} \lambda(\varpi) = \mathcal{U}_{(\alpha, -\langle \alpha, \lambda \rangle)}$ and $\lambda(\varpi) \mathcal{U}_{(-\alpha,1)} \lambda(\varpi^{-1}) = \mathcal{U}_{(-\alpha, 1-\langle \alpha, \lambda \rangle)}$. Condition (5.7) is satisfied if and only if λ is F-negative.

$\square$

Proof of Lemma 5.18. i. Let w be an F-negative element in $\tilde{\mathrm{W}}_F$. The function $f_{1,v} \,.\, \tau_w$ has support in $\mathrm{P}_F\tilde{\mathrm{I}}_F^-\hat{w}\tilde{\mathrm{I}}$. Since $\hat{w}$ satisfies (5.5), we have $\mathrm{P}_F\tilde{\mathrm{I}}_F^-\hat{w}\tilde{\mathrm{I}} = \mathrm{P}_F\hat{w}\tilde{\mathrm{I}} = \mathrm{P}_F\tilde{\mathrm{I}}$. It remains to compute the value of $f_{1,v} \,.\, \tau_w$ at 1_G (we choose 1_G as a lift for $1 \in \mathcal{D}_F$). The proof goes through exactly as in [22, 6A.3] were it is written up in the case of $\mathbf{G} = \mathrm{GL}_n$. ii. Let $d \in \mathcal{D}_F$. By Lemma 5.15i, the function $f_{1,v} \,.\, \tau_d$ has support in $\mathrm{P}_F\tilde{d}\tilde{\mathrm{I}}$. It follows from Lemma 5.15iii that its value at $\hat{d}$ is v.

∎

5.4.3. We work again with the algebraically closed field k with characteristic p. We draw corollaries from Proposition 5.13.

Corollary 5.19. *Let $F \neq x_0$ be a standard fact. If σ is an admissible k-representation of M_F with a central character, then $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}}\sigma)^{\tilde{\mathrm{I}}}$ is a finite dimensional $\tilde{\mathrm{H}}_k$-module whose irreducible subquotients are not supersingular.*

Proof. The fact that $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}}\sigma)^{\tilde{\mathrm{I}}}$ is finite dimensional is a consequence of the admissibility of σ. Let $\lambda \in \mathrm{X}_*(\mathrm{T})$ a strongly F-negative coweight (see Remark 3.6) and $\lambda_0 \in \mathrm{X}_*^+(\mathrm{T})$ the unique dominant coweight in its $\mathfrak{W}$-orbit $\mathcal{O}(\lambda)$. By Lemma 3.4

$$z_{\lambda_0} = \sum_{\lambda' \in \mathcal{O}(\lambda)} \mathcal{B}_F^-(\lambda').$$

We compute the action of z_{λ_0} on an element of the form $v \otimes 1 \in \sigma^{\tilde{\mathrm{I}}_F} \otimes_{\tilde{\mathrm{H}}_k(\mathrm{M}_F)^-} \tilde{\mathrm{H}}_k$. We have $\mathcal{B}_F^-(\lambda) = \tau_{e^\lambda}$ and therefore,

$$(v \otimes 1)\mathcal{B}_F^-(\lambda) = v \otimes \tau_{e^\lambda} = v \otimes j_F^-(\tau_{e^\lambda}^F) = (v\tau_{e^\lambda}^F) \otimes 1.$$

Recall that $\tau_{e^\lambda}^F = \tau_{\lambda(\varpi^{-1})}^F$ and that $\lambda(\varpi^{-1})$ is a central element in M_F. Therefore, $v\tau_{e^\lambda}^F = \omega(\lambda(\varpi))v$ where ω denotes the central character of σ. By (2.4), it implies that $(v \otimes 1)\mathcal{B}_F^-(\lambda') = 0$ for $\lambda' \in \mathcal{O}(\lambda)$ distinct from λ. We have proved that z_{λ_0} acts by multiplication by $\omega(\lambda(\varpi)) \neq 0$ on $\sigma^{\tilde{\mathrm{I}}_F} \otimes_{\tilde{\mathrm{H}}_k(\mathrm{M}_F)^-} \tilde{\mathrm{H}}_k$ and therefore on $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}}\sigma)^{\tilde{\mathrm{I}}}$ by Proposition 5.13. It proves the claim. ∎

Corollary 5.20. *Let Sp_F be the special k-representation of G*

$$\mathrm{Sp}_F = \frac{\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} 1}{\sum_{F' \neq F \subset \overline{F}} \mathrm{Ind}_{\mathrm{P}_{F'}}^{\mathrm{G}} 1}$$

where F' ranges over the set of standard facets $\neq F$ contained in the closure of F. The $\tilde{\mathrm{I}}$-invariant subspace of Sp_F is an irreducible $\tilde{\mathrm{H}}_k$-module which is not supersingular.

Proof. By [11, (18) and Corollary 4.3], $(\mathrm{Sp}_F)^{\tilde{\mathrm{I}}}$ is an irreducible quotient of $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} 1)^{\tilde{\mathrm{I}}}$. Apply Corollary 5.19.

∎

5.5. **On supersingular representations .** Here we suppose that $\mathfrak{F}$ is a finite extension of $\mathbb{Q}_p$. Let ρ be a weight of K. By (4.6), there is a correspondence between the k-characters of $\mathcal{H}(\mathrm{G},\rho)$ and $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ and we will use the same letter ζ for two characters paired up by (4.6). With this notation, we have a surjective morphism of representations of G:

$$\zeta \otimes_{\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)} \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1 \longrightarrow \zeta \otimes_{\mathcal{H}(\mathrm{G},\rho)} \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho. \tag{5.8}$$

For ω a character of the connected center of G, let ζ_ω the supersingular character of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ as in 5.3.3. From now on we suppose that the derived group of G is simply connected.

Lemma 5.21. *A character $\mathcal{H}(\mathrm{G},\rho)\to k$ is parametrized by the pair (G,ω) in the sense of* [17, Proposition 4.1] *if and only if it corresponds to the supersingular character ζ_ω of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$ via* (4.6).

Proof. By [17, Corollary 4.2] (see also Corollary 2.19 loc.cit), the supersingularity of the character ζ is tested on the character $\zeta' = \zeta\circ\mathfrak{T}$ and ζ is supersingular if and only if $\zeta'(\lambda)=0$ for all $\lambda\in \mathrm{X}_*^+(\mathrm{T})$ such that $\lambda(\varpi)$ does not belong to the connected center of G, which is equivalent to $\ell(e^\lambda)\neq 0$.

□

A smooth irreducible admissible k-representation of G has a central character. A smooth irreducible admissible k-representation π with central character $\omega: Z\to k^\times$ is called supersingular with respect to $(\mathrm{K},\mathrm{T},\mathrm{B})$ ([17, Definition 4.7]) if for all weights ρ of K, any map $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}\rho\to\pi$ factorizes through

$$\zeta_\omega \otimes_{\mathcal{H}(\mathrm{G},\rho)} \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho \longrightarrow \pi.$$

Note that if the first map is zero, then the condition is trivial. A supersingular representation is therefore a quotient of $\zeta_\omega \otimes_{\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)} \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1$ and of

$$\mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1/\mathfrak{I}\, \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1.$$

Remark 5.22. The representation $\mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1/\mathfrak{I}\, \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1$ depends only on the conjugacy class of x_0, and it is independant of all the choices if $\mathbf{G}$ is of adjoint type or $\mathbf{G}=\mathrm{GL}_n$ (Remark 5.10).

Theorem 5.23. *If $\mathrm{G}=\mathrm{GL}_n(\mathfrak{F})$ or $\mathrm{PGL}_n(\mathfrak{F})$, a smooth irreducible admissible k-representation π is supersingular if and only if it is a quotient of*

$$\mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1/\mathfrak{I}\, \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1.$$

Proof. Let π be a smooth irreducible admissible k-representation of G with central character ω. It is a quotient of $\zeta_\omega \otimes_{\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)} \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1$ if and only if $\pi^{\tilde{\mathrm{I}}}$ contains the supersingular character ζ_ω of $\mathcal{Z}^\circ(\tilde{\mathrm{H}}_k)$, in which case by Corollaries 5.19 and 5.20 and the main theorem of [17], π is supercuspidal and therefore supersingular.

□

The results of [17] have been generalized to the case of a $\mathfrak{F}$-split connected reductive group G in [1]. Stating a version of Theorem 5.23 in the general case would require further calculations of the $\tilde{\mathrm{H}}_k$-modules of $\tilde{\mathrm{I}}$-invariants of certain nonsupersingular representations of G. But we expect that a proof of Theorem 5.23 can be obtained independently of the classification in [17], as well as a generalization to any $\mathfrak{F}$-split group independently of the classification in [1].

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